

ON ONE REMARK ON W. FELLER'S STUDY OF THE LAW OF  
ITERATED LOGARITHM

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ABSTRACT

*The role of the extreme summands in the behavior of the entire sum in the law of the iterated logarithm is determined in this article. An example is constructed in which all the conditions of W. Feller's theorem on the law of the iterated logarithm are satisfied; however, its result does not follow. This is because of the fact that the imposed condition on the truncated variance in W. Feller's theorem is not sufficient.*

Recently, there has been a growing interest in research on the asymptotic analysis of the limiting behavior of extreme order statistics. This circumstance is due to the fact that extreme statistics serve as the basis for mathematical models of applied problems of managing economic systems, organizing the activities of financial and actuarial (insurance) companies. In this case, special attention is paid to the problems of approximating the distributions of the used statistical estimates of summation types by known infinitely divisible distributions (see, for example, [1, 2]).

In this paper, an example is given in which all W. Feller's conditions [3] are satisfied, but the result obtained by W. Feller about the influence of extreme summands on the sum in the law of the iterated logarithm does not follow. As already noted in [4], we believe that we have to impose the condition on the initial distribution of random variables more restrictive than

the similar condition  $\limsup_{n \rightarrow \infty} \frac{B(2x)}{B(x)} < \infty$  in [3], where  $B(x)$  is the truncated variance. For the sake of completeness, we present the W. Feller result.

Let  $X_1, X_2, \dots, X_n$  be symmetric, independent, identically distributed r.v. with a common distribution function  $F(x)$ .

Further, we assume that  $X_n^{(i)}$  are order statistics arranged in descending order, i.e.  $X_n^{(1)} \geq X_n^{(2)} \geq \dots \geq X_n^{(n)}$ .

Denote by

$$S_n = \sum_{i=1}^n X_i, \quad S_n^{(r)} = S_n - \sum_{i=1}^r X_n^{(i)} = \sum_{i=r+1}^n X_n^{(i)}, \quad 0 < r < n.$$



and  $\rho_n$  is the largest root of the equation

$$\rho_n^2 = nB(\rho_n), \tag{1}$$

$$B(x) = \int_{|t| \leq x} t^2 dF(t).$$

where  $|t| \leq x$  We assume that

$$I(\gamma) = \int_3^\infty \frac{x^2 dF(x)}{B(x)(\log \log x)^{1+\gamma}}, \quad \gamma \geq 0.$$

**Theorem (W. Feller).** Let the initial distribution be symmetrical,

$$\limsup_{n \rightarrow \infty} \frac{B(2x)}{B(x)} < \infty,$$

$I(0) = \infty$  and  $I(\gamma) < \infty$  for some  $\gamma \in (0, 1/4)$ . Then

$$\limsup_{n \rightarrow \infty} \frac{S_n^{(1)}}{B_n \sqrt{2 \log \log B_n}} = 1 \quad \text{a.s.,}$$

while

$$\limsup_{n \rightarrow \infty} \frac{S_n}{B_n \sqrt{2 \log \log B_n}} = \infty \quad \text{a.s.,}$$

In the following example, all the conditions of W. Feller's theorem are met, nevertheless, its statement is false.

**Example.** Let, as required in the mentioned theorem,  $B(x)$  be a monotonic function

$$B(x) \rightarrow \infty \text{ as } x \rightarrow \infty, \quad \limsup_{n \rightarrow \infty} \frac{B(2x)}{B(x)} \leq M,$$

$$\int_a^\infty \frac{x^2 dF(x)}{B(x) \log \log x} = \infty \quad \text{and} \quad \int_a^\infty \frac{x^2 dF(x)}{B(x) (\log \log x)^{1+\gamma}} < \infty \tag{2}$$

$$\text{for some } \gamma \in (0, 1/4), \text{ but } \limsup_{n \rightarrow \infty} \frac{S_n^{(1)}}{\rho_n \sqrt{\log \log \rho_n}} = \infty \quad \text{a.s.,} \tag{3}$$

where  $\rho_n$  is the largest root of equation (1).

Choosing  $0 < \varepsilon < \delta < \frac{1}{4}$ , and  $x_k$  are the solutions to equation

$$2^{(\log_2 \log_2 x_k)^{1+\varepsilon}} = 2^{k^{1+\varepsilon}} k^{1+\delta},$$

and  $k_0$  so that  $x_{k_0} < 2^{2^{k_0+1}}$ . The normalizing constant  $C_0$  is defined by the following equality

$$C_0 = 1 / \int_1^\infty \frac{g'(x) dx}{x^2}$$



We set the truncated variance it in the following form:

$$B(x) = C_0 g(x),$$

were

$$g(x) = \begin{cases} 0, & \text{if } x \in [0,1]; \\ \log_2 x, & \text{if } x \in (1,2]; \\ 2^{(\log_2 \log_2 x)^{1+\varepsilon}}, & \text{if } x \in (2, 2^{2^{k_0}}]; \\ \text{and for any } k \geq k_0 \\ \frac{2^{k^{1+\varepsilon}}}{2^{2^k}} x, & \text{if } x \in (2^{2^k}, 2^{2^k} k^{1+\delta}]; \\ \frac{2^{k^{1+\varepsilon}}}{2^{2^k}} k^{1+\delta}, & \text{if } x \in (2^{2^k} k^{1+\delta}, 2^{2^k} x_k]; \\ 2^{(\log_2 \log_2 x)^{1+\varepsilon}}, & \text{if } x \in (x_k, 2^{2^{k+1}}]. \end{cases}$$

Let us check the fulfillment of the conditions of the W. Feller theorem. Obviously,

$B(x)$  is a monotonically increasing function and  $\limsup_{n \rightarrow \infty} \frac{B(2x)}{B(x)} = 2$ . It is easy to see that for all sufficiently large  $k$ :

$$x_k \geq 2^{2^k \left(1 + \frac{\log_2 k}{2k^\varepsilon}\right)} \geq 2^{2^k} 2^{2^{k/2}}. \tag{4}$$

The graph of function  $g(x)$  has the following form:

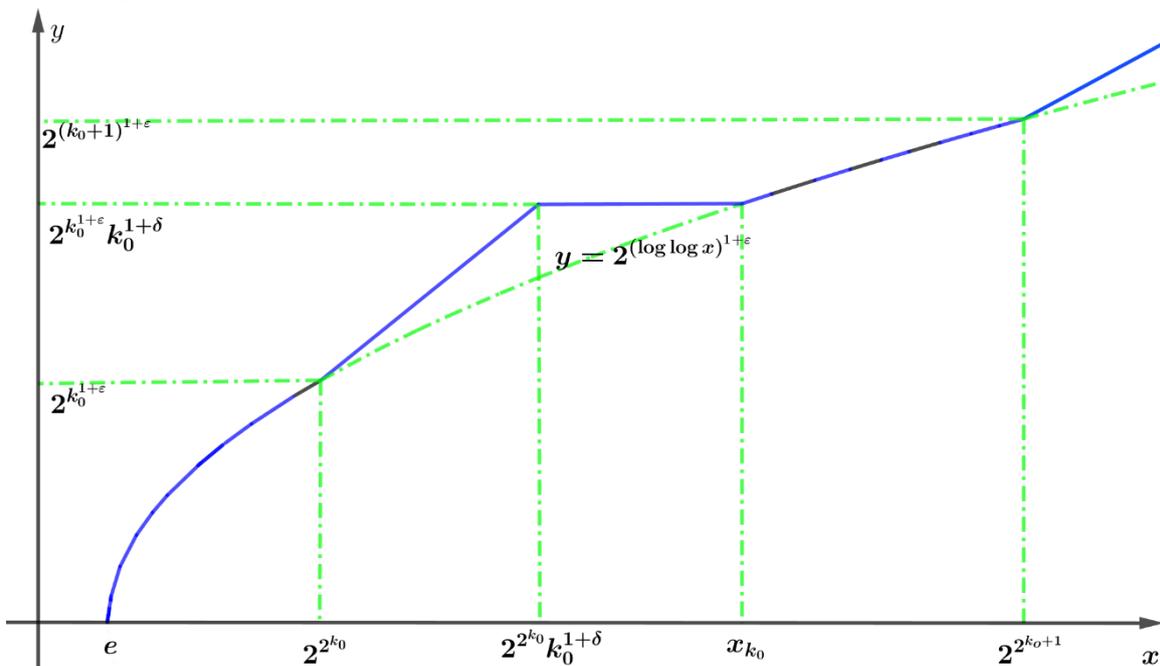


Fig.1 Function  $B(x)$  illustration.



Let us first show (2). Using the equality  $\frac{dB(x)}{dx} = 2x^2 f(x)$  and the definition of  $B(x)$ , it is easy to see that for  $k \geq k_0$  the density is:

$$f(x) = \begin{cases} C \frac{2^{k^{1+\varepsilon}}}{2^{2^k} x^2}, & \text{if } |x| \in (2^{2^k}, 2^{2^k} k^{1+\delta}]; \\ 0, & \text{if } |x| \in (2^{2^k} k^{1+\delta}, x_k]; \\ C \frac{B(x)(\log_2 \log_2 x)^\varepsilon}{x^3 \log_2 x}, & \text{if } |x| \in (x_k, 2^{2^{k+1}}]. \end{cases} \quad (5)$$

From here and from the obvious relation  $\log_2 \log_2 2^{2^k} \square \log_2 \log_2 2^{2^{k+1}}$  as  $k \rightarrow \infty$ , we obtain

$$I_1(k) = \int_{2^{2^k}}^{2^{2^k} k^{1+\delta}} \frac{x^2 dF(x)}{B(x)(\log \log x)^{1+\gamma}} \square \frac{C \log k}{k^{1+\gamma}},$$

$$I_2(k) = \int_{2^{2^k} k^{1+\delta}}^{x_k} \frac{x^2 dF(x)}{B(x)(\log \log x)^{1+\gamma}} = 0,$$

$$I_3(k) = \int_{x_k}^{2^{2^{k+1}}} \frac{x^2 dF(x)}{B(x)(\log \log x)^{1+\gamma}} \square C \int_{x_k}^{2^{2^{k+1}}} \frac{dx}{x \log x (\log \log x)^{1+\gamma+\varepsilon}}.$$

Assuming that  $A = 2^{2^{k_0}}$  we obtain

$$\int_A^\infty \frac{x^2 dF(x)}{B(x)(\log \log x)^{1+\gamma}} = \sum_{k=k_0}^\infty \sum_{i=1}^3 I_i(k) = \begin{cases} = \infty, & \text{if } \gamma = 0; \\ < \infty, & \text{if } \gamma > \varepsilon. \end{cases}$$

Thus, the validity of condition (2.1.6) is proved.

We proceed to the proof of (2.1.7). Let  $b_n$  be the largest root of the equation

$$b_n^2 = nB(b_n)(\log \log b_n)^{1+\gamma}, \quad \gamma \geq 0. \quad (6)$$

It is easy to check that  $b_n$  is a monotone sequence, such that  $b_{n+1}/b_n \rightarrow 1$  and  $b_n \rightarrow \infty$ , as  $n \rightarrow \infty$ .

Denote  $d_n = b_n$ , if  $\gamma = 0$  and  $a_n = b_n$ , if  $\gamma = \delta$ .



From the proof of Lemma 2.2.3, in particular, it follows that  $\int_A \frac{x^2 dF(x)}{B(x)(\log \log x)^{1+\gamma}}$  is

equivalent to the convergence of the series  $\sum_{n=1}^{\infty} P(b_n)$ .

From here and from (2.1.6), by the Borel–Cantelli lemma, we obtain that

$$\limsup_{n \rightarrow \infty} \frac{|\overline{X}_n^{(1)}|}{d_n} = \infty \quad \text{a.s.}, \quad \limsup_{n \rightarrow \infty} \frac{|\overline{X}_n^{(1)}|}{a_n} = 0 \quad \text{a.s.} \quad (7)$$

Next, we choose the sequences  $n_k$  and  $n'_k$  so that we have

$$a_{n_{k-1}} \leq 2^{2^k} \leq a_{n_k} \quad \text{and} \quad a_{n'_{k-1}} \leq 2^{2^k} k^{1+\delta} \leq a_{n'_k}.$$

It is obvious that  $a_{n_k} \square 2^{2^k}$  and  $a_{n'_k} \square 2^{2^k} k^{1+\delta}$ .

Hence, in view of (2.1.10) we easily obtain that

$$\rho_{n'_k} \square a_{n_k} \square 2^{2^k}, \quad n'_k \square \left(2^{2^k}\right)^2 / 2^{k^{1+\varepsilon}} \quad \text{and} \quad n_k \square \left(2^{2^k}\right)^2 / \left(2^{k^{1+\varepsilon}} k^{1+\delta}\right).$$

Let us show that relation (2.1.7) holds for the subsequence  $n'_k$ .

Let  ${}_{n_k} X_{n'_k}^{(1)} \geq {}_{n_k} X_{n'_k}^{(2)} \geq \dots \geq {}_{n_k} X_{n'_k}^{(n'_k - n_k)}$  be variational series built according to r.v.  $X_{n_k+1}, X_{n_k+2}, \dots, X_{n'_k}$ .

$${}_{n_k} S_{n'_k}^{(1)} = \sum_{i=n_k+1}^{n'_k} X_i - {}_{n_k} X_{n'_k}^{(1)}, \quad z_k = \rho_{n'_k} \sqrt{\log \log \rho_{n'_k}}.$$

To show (2.1.7), it suffices to prove from (2.1.11) that

$$\limsup_{k \rightarrow \infty} \frac{{}_{n_k} S_{n'_k}^{(1)}}{z_k} = \infty \quad \text{a.s.} \quad (8)$$

and 
$$\limsup_{k \rightarrow \infty} \frac{|S_{n_k}|}{z_k} = 0 \quad \text{a.s.} \quad (9)$$

Since  $\left\{ {}_{n_k} S_{n'_k}^{(1)} > a \right\}$  is a sequence of pairwise incompatible events, then, by virtue of the Borel–Cantelli lemma, to prove (2.1.12), it suffices to show that

$$\sum_{k=k_0}^{\infty} P\left( {}_{n_k} S_{n'_k}^{(1)} \geq \tau z_k \right) = \infty \quad (10)$$

for any  $\tau > 0$ . On the other hand, it is easy to obtain from (2.1.9) that as  $k \rightarrow \infty$

$$P(a_{n_k}) \square \int_{a_{n_k}}^{a_{n_k}'} f(x) dx \square \frac{2^{k^{1+\varepsilon}}}{(2^{2^k})^2}, \tag{11}$$

for any  $\tau > 0$ . For all sufficiently large  $k$ , we have

$$P\left( S_{n_k}^{(1)} \geq \tau z_k \right) \geq P \left( \bigcup_{\substack{n_k \leq i, j \leq n_k' \\ i \neq j}} \left[ X_i \geq \tau z_k, X_j \geq \tau z_k, \sum_{\substack{m=n_k+1 \\ m \neq i, j}}^{n_k'} X_m \geq 0, X_l < \tau z_k, n_k < l \leq n_k', l \neq i, j \right] \right) \geq C_{n_k' - n_k}^2 P^2(\tau z_k) \left[ P(S_{n_k' - n_k - 2} \geq 0) - n_k' P(\tau z_k) \right] \geq \geq C [n_k' P(\tau z_k)]^2 \geq \frac{C}{k}.$$

This implies (2.1.14). Thus, (2.1.12) is proved.

To prove (2.1.13), it suffices to show that

$$\sum_{k=k_0}^{\infty} P(|S_{n_k}| \geq \tau z_k) < \infty \tag{12}$$

for any  $\tau > 0$ .

We have

$$\sum_{k=k_0}^{\infty} P(|S_{n_k}| \geq \tau z_k) \leq \sum_{k=k_0}^{\infty} P(|\bar{X}_{n_k}^{(1)}| \geq a_{n_k}) + \sum_{k=k_0}^{\infty} P(|S_{n_k}| \geq \tau z_k, |\bar{X}_{n_k}^{(1)}| < a_{n_k}) = \sum_1 + \sum_2$$

Since  $\sum_{k=k_0}^{\infty} P(|\bar{X}_{n_k}^{(1)}| \geq a_{n_k}) \leq 2n_k P(a_{n_k})$ , then in view of (2.1.15)  $\sum_1$  converges.

Next, using the Chebyshev inequality, we obtain the following for all sufficiently large  $k$

$$P\left(|S_{n_k}| \geq \tau z_k, |\bar{X}_{n_k}^{(1)}| < a_{n_k}\right) \leq P\left(\left[\sum_{i=1}^n X_i I(|X_i| < a_{n_k})\right]^2 \geq \tau^2 z_k^2\right) \leq \frac{n_k B(a_{n_k})}{\tau^2 z_k^2} \leq \frac{C}{k^{2+\delta}}.$$

$$\sum_{k=k_0}^{\infty} P\left(|S_{n_k}| \geq \tau z_k, |\bar{X}_{n_k}^{(1)}| < a_{n_k}\right) \leq C \sum_{k=k_0}^{\infty} \frac{1}{k^{2+\delta}}.$$

So, the series  $\sum_2$  also converges.

Relation (2.1.16) is proved.

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