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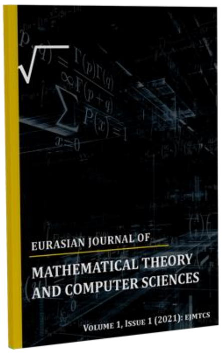

ILL-POSED PROBLEMS AND EXAMPLES OF INCORRECT PROBLEMS

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ABSTRACT

This work examines the problem of ill-posed problems, which often arise in various fields of science and practice. Ill-posed problems are characterized by the lack of a unique solution, sensitivity to changes in input data, and instability of the solution. The aim of the work is to study the essence of ill-posed problems, identify the reasons for their occurrence, and consider examples from different areas of knowledge. Particular attention is paid to the theoretical aspects of ill-posed problems, such as non-uniqueness of the solution, sensitivity to input data, and uncertainty. The paper describes methods for solving ill-posed problems, including regularization, optimization, statistical, and machine learning methods. These approaches allow to deal with uncertainty and instability of the solution. The study of ill-posed problems is of great practical importance, as they often occur in real-life situations and require special methods for their solution. This research will be useful for specialists in various fields, as well as for anyone who faces the task of solving problems in their daily activities.

Introduction

In the modern world, problem solving is an integral part of our daily life and scientific research. However, not all tasks can be considered correctly posed. Ill-posed problems, which either have an infinite number of solutions or have no solutions at all, play an important role in various fields of science and practice.

Ill-posed problems are a class of problems for which the formulation does not provide an unambiguous solution. They can arise due to various reasons, such as insufficient information, uncertainty in data, noise or measurement errors. The main characteristics of ill-posed problems include their sensitivity to changes in input data and the lack of a stable solution.

The purpose of this course work is to study the essence of incorrectly posed problems, identify the reasons for their occurrence, and also consider examples of such problems from various fields of knowledge.

As part of the study, we will consider both the theoretical aspects of ill-posed problems and their practical application. We will give examples of ill-posed problems from mathematics, physics, economics, information technology and other fields to illustrate the variety of their manifestations.

The main objective of the work is to identify methods for solving incorrectly posed problems and evaluate their effectiveness. We will also try to identify possible ways to prevent the occurrence of incorrect tasks in the future.

The study of this topic is of practical importance for specialists in various fields of knowledge, as well as for everyone who is faced with solving problems in their daily activities.

Theoretical aspects of ill-posed problems

Incorrectly posed task- a problem that does not have any of the properties of a well-posed problem.

Examples of typical well-posed problems are Dirichlet problem For Laplace equations And diffusion equation with given initial conditions. They can be considered as "natural" problems - in the sense that there are physical processes described by solutions to these problems. On the other hand, the inverse problem for the diffusion equation - finding the previous temperature distribution from the final data - is not well posed, because its solution is very sensitive to changes in the final data.

Very often it turns out that the inverse problems. Continuous tasks like this are often discretizeto obtain a numerical solution. Despite the fact that from the point of view functional analysis Such problems are usually continuous, they can be subject to instability of the numerical solution when calculating with finite accuracy or when there are errors in the data. Incorrect tasks may occur during processing geophysical, geological, astronomical observations, problem solving optimal control and planning.

Even if the problem is correctly posed, it can still be ill-conditioned, that is, a small error in the initial data can lead to much larger errors in solutions. Ill-conditioned tasks are characterized by greater condition number.

If the problem is formulated correctly, then there is a good chance of solving it numerically usingstable algorithm. If the task is formulated incorrectly, then its formulation needs to be changed; This usually involves introducing some additional assumptions (such as the assumption that the solution is smooth). This procedure is called regularization, and the most widely used Tikhonov regularization, applicable to linear ill-posed problems.

The main characteristics of ill-posed problems include:

1. **Non-uniqueness of the solution**: Ill-posed problems may have an infinite number of solutions or none at all.
2. **Input Sensitivity**: Small changes in input data can lead to significant changes in the solution of the problem.
3. **Uncertainty**: Input data may be incomplete or noisy, which introduces uncertainty into the problem solving process.

Various methods are used to solve ill-posed problems, including regularization methods, optimization methods, statistical methods, and machine learning methods. These methods make it possible to cope with the uncertainty and instability of the problem solution.

Studying the theoretical aspects of ill-posed problems allows us to understand the nature of their occurrence, features and methods of solving them. This is important for scientific research and practical applications, since ill-posed problems often occur in real situations and require a special approach to solve them.

Historical facts and basic mathematical definitions of the theory of well-posed and ill-posed problems

The concept of a "well-posed problem" was first introduced by J. Hadamard in 1923 and related only to boundary value problems of mathematical physics. The correctness of the problem statement was ensured by the fulfillment of two conditions: the existence of a solution and its uniqueness. The requirement for solution stability was subsequently added to the first two. For a long time, according to the authoritative opinion of J. Hadamard, it was believed that ill-posed problems cannot have practical meaning and therefore there is no need to solve them. The first work in which this opinion was refuted is considered to be the famous work of Academician A. N. Tikhonov in 1943, in which he first formulated a conditionally correct problem and solved one of the pressing problems of exploration geophysics. Subsequently, the theory of ill-posed problems took its well-deserved place in scientific research, since it was repeatedly confirmed that problems arising in practice are most often ill-posed, and the mathematical apparatus for solving them is precisely the theory of correct and ill-posed problems by A. N. Tikhonov. Active research work on this topic is currently underway, and achievements have been recognized.

Mathematical problems most often consist of searching for a solution z using initial data u. In this case, it is assumed that u and z are connected by the dependence

$z = R(u)$. A problem is called correct or correctly posed if the following conditions are met:

1) the problem has a solution for any admissible initial data u (existence of a solution);
2) each initial data u corresponds to only one solution z (uniqueness of the solution);
3) the solution is stable.

The meaning of the first condition is that there are no contradictory conditions among the initial data, which would exclude the possibility of solving the problem. The second condition means that the initial data is sufficient for the unique solvability of the problem. These two conditions are usually called the conditions for the mathematical certainty of the problem. The third condition is the following. If u_1 and u_2 are two different sets of initial data, the measure of deviation of which from each other is sufficiently small, then the measure of deviation of the solutions $z_1 = R(u_1)$ and $z_2 = R(u_2)$ is less than any predetermined accuracy. It is assumed that in the variety $U = \{u\}$ of admissible initial data and in the variety of possible solutions $Z = \{z\}$ the concept of deviation measure $\rho u\ (u_1, u_2)$ and $\rho z\ (z_1, z_2)$ is established.

The third condition is interpreted as the physical certainty of the problem. If u_1 and u_2 are two different sets of initial data, the measure of deviation of which from each other is sufficiently small, then the measure of deviation of the solutions $z_1 = R(u_1)$ and $z_2 = R(u_2)$ is less than any predetermined accuracy. It is assumed that in the variety $U = \{u\}$ of admissible initial data and in the variety of possible solutions $Z = \{z\}$ the concept of deviation measure $\rho u\ (u_1, u_2)$ and $\rho z\ (z_1, z_2)$ is established. The third condition is interpreted as the physical certainty of the problem. This is explained by the fact that the initial data of a physical problem are, as a rule, specified with some error; if the third condition is violated, any small perturbations of the initial data can cause large deviations in the solution. Problems that do not satisfy at least one of the correctness conditions are called incorrect or incorrectly posed.

Incorrectly set tasks

The operator equation is considered as the main object:

$Az=u$,

where A is a linear operator acting from the Hilbert space Z to the Hilbert space U. It is required to find a solution to the operator equation z corresponding to a given inhomogeneity (or right-hand side of the equation) u.

Such an equation is a typical mathematical model for many physical, so-called inverse, problems, if we assume that the desired physical characteristics z cannot be directly measured, and as a result of the experiment only data u associated with z using the operator A can be obtained.

The French mathematician J. Hadamard formulated the following conditions for the correctness of the formulation of mathematical problems, which we will consider using the example of a written operator equation. The problem of solving an operator equation is called well-posed (according to Hadamard) if the following three conditions (correctness conditions) are satisfied:

1) the problem has a solution for any admissible initial data (a solution exists$\forall u \in U$);
2) each initial data u corresponds to only one solution (the solution is unique);
3) the solution is stable (if $u_n \to u$, $Az_n=u_n=u$, then $z_n \to z$).

The meaning of the first condition is that there are no contradictory conditions among the initial data, which would exclude the possibility of solving the problem.

The second condition means that the initial data is sufficient for an unambiguous solution to the problem. These two conditions are usually called the conditions for the mathematical certainty of the problem. Condition 2) A^{-1} is satisfied if and only if the operator A is one-to-one (injective). Conditions 1) and 2) mean that there is an inverse operator, and its domain of definition D(A^{-1}) (or the set of values of the operator A, R(A)) coincides with U.

Condition 3) means that the inverse operator A^{-1} is continuous, i.e. “small” changes in the right-hand side u correspond to “small” changes in the solution z. The third condition is usually interpreted as the physical determinism of the problem. This is explained by the fact that the initial data of a physical problem are, as a rule, specified with some error; if the third condition is violated, arbitrarily small perturbations of the initial data can cause large deviations in the solution.

Problems that do not satisfy at least one correctness condition are called ill-posed problems (or incorrectly posed). Moreover, J. Hadamard believed that only well-posed problems should be considered when solving practical problems. However, examples of ill-posed problems are well known, the study and numerical solution of which must be resorted to when considering numerous applied problems. It should be noted that the stability and instability of a solution are related to how the solution space Z is defined. The choice of the solution space (including the norms in it) is usually determined by the requirements of the applied problem. Problems may be incorrectly posed with one choice of norm and correctly posed with another.

Numerous inverse (including ill-posed) problems can be found in various fields of physics. Thus, an astrophysicist cannot actively influence the processes occurring on distant stars and galaxies; he has to draw conclusions about the physical characteristics of very distant objects based on their indirect manifestations, available measurements on Earth or near the Earth (at space stations). Excellent examples of ill-posed problems can be found in medicine; first of all, computational (or computed tomography) should be noted. The applications of ill-posed problems in geophysics are well known (in fact, it is easier and cheaper to judge what is happening under the Earth's surface by solving inverse problems than to drill deep wells), radio astronomy, spectroscopy, nuclear physics, etc.

A well-known example of an ill-posed problem is the Fredholm integral equation of the 1st kind. Let operator A have the form:

$$Az \equiv \int_a^b K(x,s)\, z(s)ds = \sim(x), x \in [c,d].$$

Let the kernel of the integral operator K(x, s)- a function continuous over the set of arguments x∈[c,d], s∈[a,b], and the solution z(s)- continuous on a segment[a,b]function. Thus, we can consider the operator A as acting in the following spaces: A :C[a,b]→C[c,d]. (Space C[a,b]consists of functions that are continuous on the interval[a,b]. Norm z∈C[a,b]is defined as). Let us show that in this case the problem of solving the integral equation is ill-posed. To do this, you need to check the conditions for the correctness of the problem statement:$\|z\|_{C[a,b]} = \max_{s\in[a,b]} |z(s)|$

1) Existence of a solution for any continuous line[c,d]functions u(x). In fact, this is not so: there are infinitely many continuous functions for which there is no solution.
2) Uniqueness of the solution. This condition is satisfied if and only if the kernel of the integral operator is closed.

The first two correctness conditions are equivalent to the condition for the existence of an inverse operator with the domain of definition D()=C$A^{-1}A^{-1}$[c,d]. If the kernel of an integral operator is closed, then the inverse operator exists, but its domain of definition does not coincide with C[c,d].

3) Stability of the solution. This means that for any sequence the sequence zn$u_n \to \ \bar{u}\ (Az_n = u_n, \bar{z} = \bar{u})$→$\bar{z}$. Stability is equivalent to continuity inverse operator provided that the inverse operator exists. In this case it is notA^{-1} so, as can be seen from the following example. Let a sequence of continuous functions , n=1, 2, ..., such that on the interval and vanishes outside the given interval, max| z (s) |=1, s$z_n(s) z_n(s) \neq 0 \left[\frac{a+b}{2} - d_n, \frac{a+b}{2} + \ d_n\right]$∈[a, b], and the sequence of numbers d→0+0 .
4) Such a function can be chosen, for example, piecewise linear. Then for any x∈[c,d]

$$|u_n(x)| = \left|\int_a^b K(x,s) z_n(s)ds\right| = \left|\int_{\frac{a+b}{2}-d}^{\frac{a+b}{2}+d} K(x,s) z_n(s)ds\right| \leq K_0 \cdot 1 \cdot 2d_n \to 0$$

at$n \to \infty$, где $K_0 = max|K(x,s)|\,, x \in [c,d], s \in [a,b]$.

The sequence of functions is uniform, i.e. by norm C[c, d], converges to$u_n(x)\bar{u}$ =0.

Although the solution to the equation in this case$A\bar{z} = \bar{u}\bar{z}$=0 , the sequence does not tend to , since.$z_n\bar{z}\|z_n - \bar{z}\|_{C[a,b]} = 1$

The integral operator A is completely continuous when acting from to$L_2[a,b]$

$L_2[c,d]$, when acting from C[a, b] to and when acting from C[a, b] to C[c, d]. (Space$L_2[c,d]$

$L_2[a,b]$consists of functions that are square integrable on the interval[a,b]. Norm z∈$L_2[a,b]$ defined as$\|z\|_{L_2[a,b]} = \left\{\int_a^b z^2(s)ds\right\}^{1/2}$). This means that any limited This operator converts the sequence into a compact one. Compact sequence by definition has the property that from any of its subsequences one can isolate convergent. It is easy to indicate a sequence , , from which it is impossible to select a subsequence converging in C[a, b]. For example,$z_n\|z_n\|_{L_2[a,b]} = 1$

$$z_n(x) = \left(\frac{2}{b-a}\right)^{1/2} \sin\frac{\pi n(x-a)}{b}; n = 1,2, \dots..$$

The norms of all terms of this sequence are equal to 1 in , but it is impossible to isolate a convergent subsequence from any subsequence of this sequence, since . Obviously, this sequence consists of functions continuous on [a, b] and is uniformly (in the norm of C[a, b]) bounded, but from this sequence it is impossible to isolate a subsequence converging in C[a, b] (then it would converge and in , since uniform convergence implies convergence in average). If we assume that the operator is continuous, then it is easy to come to a contradiction. For the inverse operator to exist, it suffices to require that the forward operator A be injective. Obviously, if operator B: C[c, d] $L_2[a,b]\|z_i - z_j\| = \sqrt{2}, i \neq j$ $L_2[a,b]$ $A^{-1}\rightarrow C$[a, b] is continuous, and the operator A is completely continuous, then BA :C[a, b]→C[a, b] is also a completely continuous operator. But then, since for any n, the sequence is compact, which is not true. An operator that is the inverse of a completely continuous operator cannot be continuous. A similar proof can be carried out for any infinite-dimensional Banach (i.e., complete normed) spaces.$A^{-1}Az_n = z_n z_n$

Since the problem of solving the Fredholm integral equation of the first kind in the indicated spaces is formulated incorrectly, even with very small errors in specifying u(x), the solution may either be absent or be very different from the desired exact solution.

So, a completely continuous injective operator has an inverse operator, which is not continuous (bounded). Moreover, when acting in infinite-dimensional Banach spaces, the set of values of a completely continuous operator is not closed. Therefore, no matter how close you want to the inhomogeneity u(x), for which a solution to the operator equation exists, there will be an inhomogeneity for which there is no solution.

Incorrect formulation of a mathematical problem may be due to an error in the operator’s assignment. The simplest example is given by the problem of finding a normal pseudo-solution to a system of linear algebraic equations and the resulting instability associated with errors in specifying the matrix.

Method for selecting solutions to ill-posed problems

The ability to determine approximate solutions to ill-posed problems that are resistant to small changes in the initial data is based on the use of additional information regarding the solution. Various types of additional information are possible.

In the first category of cases, additional information of a quantitative nature makes it possible to narrow the class of possible solutions, for example, to a compact set, and the problem becomes resistant to small changes in the initial data. In the second category of cases, to find approximate solutions that are stable to small changes in the initial data, only qualitative information about the solution is used (for example, information about the nature of its smoothness).

As an ill-posed problem, we will consider the problem of solving the equation

$Az=u$ (1)

relative to z, where $u \in U$, $z \in F$, U and F are metric spaces. The operator A maps F onto U. It is assumed that there is an inverse operator A-1, but it is not, generally speaking, continuous.

An equation with an operator A possessing the indicated properties will be called an operator equation of the first kind, or, in short, an equation of the first kind.

A widely used method in computing practice for approximate solution of an equation is the selection method. It consists in the fact that for elements z of some predetermined subclass of possible solutions M (M ∈ F), the operator Az is calculated, i.e., the direct problem is solved. As an approximate solution, we take an element z0 from the set M at which the discrepancy ρU(Az, u) reaches a minimum, i.e.

$\rho U(Az_0, u)=\inf \rho U(Az, u)$

$z \in M$

Let the right side of equation (1) be known exactly, i.e. u=uT, and it is required to find its solution zT. Usually, M is taken to be a set of elements z, depending on a finite number of parameters that vary within limited limits so that M is a closed set of a finite-dimensional space. If the desired exact solution zT of equation (1) belongs to the set M, then $\inf_{z \in M} \rho_U(Az,u)=0$ and this lower bound is achieved at the exact solution zT. If equation (1) has a unique solution, then the element z0 that minimizes ρU(Az,u) is uniquely determined.

In practice, the minimization of the residual ρU(Az, u) is carried out approximately and the following important question arises about the effectiveness of the selection method, i.e. about the possibility of getting as close as you like to the desired exact solution.

Let {zn} be a sequence of elements for which ρU(Azn, u)→0 as n→∞. Under what conditions can we assert that $\rho F(z_n, z_T) \to 0$, i.e., that {zn} converges to z_T?

This is a question of justifying the effectiveness of the selection method.

The desire to justify the success of the selection method led to the establishment of general functional requirements that limit the class of possible solutions M for which the selection method is stable and $z_n \to z_T$. These requirements consist in the compactness of the set M and are based on the well-known topological lemma given below.

Lemma.*Let the metric space F be mapped onto the metric space U and Uo is the image of the set Fo, Fo∈ F, with this display. If the mapping F∈U is continuous, one-to-one and the set Fo is compact on F, then the inverse mapping Uo→F other set Uo onto the set Fo is also continuous in the metric of the space F.*

Proof.Let z be elements of the set F ($z\in F$), and u are elements of the set U ($u\in U$). Let function u=?(z)performs a direct mapping of $F\rightarrow U$, and the function z=?(u)—inverse mapping $U\rightarrow F$.

Notes and comments:Tikhonov's method for solving ill-posed problems

Tikhonov's method, also known as the regularization method, is used to solve ill-posed problems. This is a class of problems that may either have no solution, have an infinite number of solutions, or the solutions may be unstable to small changes in the data or conditions of the problem. Tikhonov's method helps to make these problems stable and to find solutions that best fit the data.

The idea of Tikhonov's method is to add a regularizer to the objective function, which is usually a combination of the squared norm of the model parameters and the squared norm of the difference between the solution and the observed data. This approach allows you to find solutions that approximate the data well, but are not too sensitive to noise or small changes in the data.

Suppose we have an ill-posed problem in the form of a linear equation Ax=y, where A is the operator, x is the unknown vector we want to find, and y is the observed data. However, due to noise or insufficient information, this equation may be incorrectly defined or not robust to small changes in the data. To deal with this problem, we can use Tikhonov's method.

Tikhonov's method is formulated through minimizing the error functional, which consists of two parts: the approximation error and the regularization term. Typically the quadratic error function and the squared solution norm are used:

$$\Phi(x) = \|Ax - y\|^2 + \alpha\|x\|^2$$

Here the first term$\|Ax-y\|^2$measures the approximation error between predicted data$Ax$and observed data y, and the second term$\|x\|^2$is a regularization term that controls the complexity of the solution by penalizing large values of x. The parameter α determines the weight of the regularization term.

Thus, the task is to find a solution x that minimizes the error functional Φ(x). This can be done by solving the equation:

$$\nabla\Phi(x) = 0$$

Where $\nabla\Phi(x)$- gradient of the error functional in x.

Solving this equation will give the optimal value of x, which takes into account both the accuracy of the data approximation and the regularization to stabilize the solution.

Conclusion

In conclusion, we can summarize that ill-posed problems represent an important class of problems that require special solution methods. They arise in various fields, ranging from science and engineering to medicine and information technology. These problems have certain features, such as a lack of information, the presence of noise, or instability to small changes in data, which makes their solution non-trivial.

Examples of incorrectly posed problems may include image restoration, inverse problems in physics and medicine, signal decoding, and many others. To solve them, regularization methods, such as Tikhonov's method, as well as other techniques, for example, optimization methods or machine learning algorithms, are often used.

Understanding and developing methods for solving ill-posed problems play a key role in scientific and engineering progress, opening up new opportunities for data analysis, information recovery and informed decision-making in various areas of human activity.

Depending on the level of task complexity, the decision environment varies according to the degree of risk. Conditions of certainty exist when the manager knows exactly the outcome that each choice will have.

Methods for approximate solution of ill-posed problems and their applications to solving inverse problems are important for automating the processing of observations and for solving control problems. There are many works (especially by Soviet mathematicians) devoted to these methods.

There was an opinion that ill-posed problems cannot be encountered when solving physical and technical problems and that for ill-posed problems it is impossible to construct an approximate solution in the absence of stability. The expansion of automation tools in obtaining experimental data has led to a large increase in the volume of such data; the need to establish information about natural scientific objects from them required the consideration of ill-posed problems. The development of electronic computer technology and its application to solving mathematical problems has changed the point of view on the possibility of constructing approximate solutions to ill-posed problems.

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