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BIR JINSLI BO'LMAGAN KASR TARTIBLI TENGLAMA UCHUN KOSHI MASALASI YECHIMI

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KALIT SO'ZLAR
Koshi masalasi, Kaputo ma'nosidagi kasr tartibli hosila, Mittag–Leffler funksiyasi, Dirakning delta funksiyasi.

ANNOTATSIYA
Ushbu ishda kasr tartibli oddiy differensial tenglama uchun qo'yilgan Koshi masalasini Laplas hamda teskari Laplas almashtirishlari yordamida yechish masalasi hamda bu yechimning klassik yechim bo'lish masalasi o'rganilgan.

Masalaning qo'yilishi: Bizga quyidagi bir jinsli bo'lmagan kasr tartibli differensial tenglama uchun Koshi masalasi berilgan bo'lsin:

$$\left[\begin{array}{c} D_{0+}^{\alpha}y(x)+\lambda y(x)=f(x) \\ y(0)=y_0 \end{array}\right., x>0, 0<\alpha<1$$

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bu yerda

$$D_{0+}^{\alpha}y(x)=?$$

(1) tenglamaning yechimi mavjud deb, uning yechimini topamiz. Buning uchun $D_{0t}^{\alpha}y(x)$ Kaputo ma'nosidagi kasr tartibli hosila uchun $\lim\limits_{\alpha\to+0} D_{0+}^{\alpha}y(x)$ va $\lim\limits_{\alpha\to1-0} D_{0+}^{\alpha}y(x)$ limitlarni mavjudligini ko'rsatamiz.

1) $\lim\limits_{\alpha\to+0} D_{0+}^{\alpha}y(x)=\lim\limits_{\alpha\to+0}\frac{1}{\Gamma(1-\alpha)}\int_0^x\frac{y'(t)dt}{(x-t)^{\alpha}}=\frac{1}{\Gamma(1)}\int_0^x y'(t)dt=\int_0^x y'(t)dt=y(t)\Big|\begin{matrix} t=x \\ t=0\end{matrix}=y(x)-y(0)=y(x)-y_0$

$$\lim_{\alpha\to+0} D_{0+}^{\alpha}y(x)=y(x)-y_0$$

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Yuqoridagi $\int_0^x\frac{y'(t)dt}{(x-t)^{\alpha}}$ integral regulyarizatsilangan integral bo'lganligi uchun to'g'ridan to'g'ri limitga o'tamiz. Natijada

2) $\lim\limits_{\alpha\to1-0} D_{0+}^{\alpha}y(x)=\lim\limits_{\alpha\to1-0}\frac{1}{\Gamma(1-\alpha)}\int_0^x\frac{y'(t)dt}{(x-t)^{\alpha}}=\frac{1}{\Gamma(0)}\int_0^x\delta(x-t)y'(t)dt=y'(x)$

$$\lim_{\alpha\to1-0} D_{0+}^{\alpha}y(x)=y'(x)=\lim_{x\to x_0}\frac{y(x)-y(x_0)}{x-x_0}$$

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(3) ifodadan ko'ramizki $\alpha\to1-0$ da klassik hosilani berar ekan.

Bu yerda

$\frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)}\to\delta(x-t),\ \ (\alpha\to1)$ - Adamr yadrosi,

$\delta(x-x_0)=\begin{cases}0, & x\neq x_0\\ \infty, & x=x_0\end{cases}\wedge\int_{-\infty}^{+\infty}\delta(x-x_0)dx=1$ – Dirakning deltasi

$\forall\varphi\in D(R)=C_0^{\infty}R,\quad \int_{-\infty}^{+\infty}\delta(x-x_0)\varphi(x)dx=\varphi(x_0),\ \ \delta(x)=\delta(-x)$

Koshi masalasi (1) ni yechish 2 ta holatda qarab chiqiladi:

1. $\alpha \to +0$ holatda (2) ifodani (1) qo'ysak $y(x) - y_0 + \lambda y(x) = f(x)$ bundan esa

$$y(x) = \frac{1}{1+\lambda}[f(x) + y_0], \quad \lambda \neq -1 \qquad 4$$

yechim hosil bo'ladi.

2. $\alpha \to 1-0$ holat uchun (3) ifodani (1) ga qo'ysak

$$\begin{cases} y'(x) + \lambda y(x) = f(x) \\ y(0) = y_0 \end{cases} \qquad 5$$

(5) tenglama klassik ma'nodagi bir jinsli bo'lmagan Koshi masalasi bilan bir xil ko'rinishga kelib qoldi. Bu tenglamani Laplas almashtirishi yordamida yechilsa quyidagi hosil bo'ladi

$$y(x) = y_0 \int e^{-\lambda(x-t)} f(t)dt \qquad 6$$

Yuqorida biz $\alpha \to +0$ va $\alpha \to 1-0$ bo'lgan ikkita holatda (1) bir jinsli bo'lmagan kasr tartibli Koshi masalasining yechimini qarab chiqdik.

Keyingi navbatda esa (1) bir jinsli bo'lmagan kasr tartibli differensial tenglama uchun Koshi masalasining umumiy yechimini topish ketma – ketligini qarab chiqamiz. Buning uchun Laplas almashtirishlaridan foydalanamiz.

$y(x)$ ning Laplas almashtirishi:

$$L\{y(x)\}(s) = \int_0^\infty e^{-sx}\, y(x)dx; \quad Res > Res_0; \ |y(x)| \leq Me^{s_0 x}; \quad M > 0 \quad (7)$$

$y'(x)$ ning Laplas almashtirishi:

$$L\{y'(x)\}(s) = \int_0^\infty e^{-sx}\, y'(x) = e^{-sx}y(x)|_0^\infty + s\int_0^\infty e^{-sx}y(x)dx \qquad 8$$

$Res > Res_0$; $|y(x)| \leq Me^{s_0 x}$; $M > 0$ shartlarga ko'ra

$$\lim_{x\to\infty} e^{-sx}y(x) \leq \lim_{x\to\infty} e^{-sx}\cdot M \cdot e^{s_0 x} = M\lim_{x\to\infty} e^{-(s-s_0)x} = 0 \qquad 9$$

(9) ifodaga ko'ra (8) ifoda quyidagi ko'rinishga keladi:

$$L\{y'(x)\}(s) = 0 - y(0) + s\int_0^\infty e^{-sx}y(x)dx = sL\{y\}(s) - y(0) \qquad 10$$

(10) Laplas almashitirishidan kelib chiqib $D_{0+}^{\alpha}y$ ning Laplas almashtirishini yozamiz

$$D_{0+}^{\alpha}y \ : \quad L\{D_{0+}^{\alpha}y\}(s) = s^{\alpha}L\{y\}(s) - y(0)s^{\alpha-1}. \qquad 11$$

Yuqoridagi Laplas almashtirishlarini (1) tenglamaga qo'yamiz

$$s^{\alpha}L\{y(x)\}(s) - y_0 s^{\alpha-1} + \lambda L\{y(x)\}(s) = L\{f(x)\}(s). \qquad 12$$

Budan

$$L\{y(x)\}(s) = \frac{s^{\alpha-1}}{s^{\alpha}+\lambda}y_0 + \frac{1}{s^{\alpha}+\lambda}L\{f(x)\}(s) \qquad 13$$

ifoda hosil qilamiz va teskari Laplas almashtirishini almashtirishlarini qo'llaymiz.

$$\frac{s^{\beta-\gamma}}{s^{\beta}\pm\omega} \risingdotseq t^{\gamma-1}E_{\beta,\gamma}(\mp\omega t^{\beta}) \quad \text{- Teskari Laplas almashtirishi.} \qquad 14$$

Bu yerda $E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}$ – Mittag Lefler funksiyasi. (14) ga ko'ra

$$\frac{s^{\alpha-1}}{s^{\alpha}+\lambda} \risingdotseq x^{1-1}E_{\alpha,1}(-\lambda x^{\alpha}) = E_{\alpha}(-\lambda x^{\alpha})$$

$$\frac{1}{s^{\alpha}+\lambda} \risingdotseq x^{\alpha-1}E_{\alpha,\alpha}(-\lambda x^{\alpha})$$

Yuqoridagi teskari Laplas almashtirishlariga ko'ra hamda (13) yechim quyidagi ko'rinishga keladi:

$$L\{y(x)\}(s) = L\{E_{\alpha}(-\lambda x^{\alpha})\}(s)\cdot y_0 + L\{x^{\alpha-1}E_{\alpha,\alpha}(-\lambda x^{\alpha})\}(s)\cdot L\{f(x)\}(s); \quad 15$$

1. $L^{-1}L = LL^{-1} = I$
2. $L\{f\}\cdot L\{g\} = L\{f * g\} = L\{\int_0^x f(x-y)\cdot g(y)dy\,(s)\}$ xossalarga ko'ra

$$y(x) = E_{\alpha}(-\lambda x^{\alpha})\cdot y_0 + \int_0^x y^{\alpha-1}E_{\alpha,\alpha}(-\lambda y^{\alpha})f(x-y)dy; \qquad 16$$

yechim hosil bo'ladi.

Keyingi navbatda (16) yechimning klassik yechim ekanligini ko'rsatishimiz kerak. Buning uchun (4) va (6) yechimlarni (16) yechim bilan ustma ust tushishini ko'rsatishimiz kerak bo'ladi.

Birinchi navbatda (16) yechimni Mittag Lefler funksiyasini qo‘llagan holda soddalashtirib olamiz.

$y(x) = y_0 \sum_{n=0}^{\infty} \frac{(-\lambda x^{\alpha})^n}{\Gamma(\alpha n+1)} + \int_0^x y_0{}^{\alpha-1} \sum_{n=0}^{\infty} \frac{(-\lambda y^{\alpha})^n}{\Gamma(\alpha n+\alpha)} f(x-y)dy = I_1 + I_2$

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$I_1 = y_0 \sum_{n=0}^{\infty} \frac{(-\lambda x^{\alpha})^n}{\Gamma(\alpha n+1)}$ 18

$I_2 = \int_0^x y_0{}^{\alpha-1} \sum_{n=0}^{\infty} \frac{(-\lambda y^{\alpha})^n}{\Gamma(\alpha n+\alpha)} f(x-y)dy$

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$\lim_{\alpha\to+0} I_1 = \lim_{\alpha\to+0} \left(y_0 \sum_{n=0}^{\infty} \frac{(-\lambda x^{\alpha})^n}{\Gamma(\alpha n+1)}\right) = y_0 \sum_{n=0}^{\infty} (-\lambda)^n = \frac{1}{1+\lambda} y_0;\ |\lambda| < 1$ 20

$$\lim_{\alpha\to+0} I_2 = \lim_{\alpha\to 1-0} (\int_0^x y_0{}^{\alpha-1} \sum_{n=0}^{\infty} \frac{(-\lambda y^{\alpha})^n}{\Gamma(\alpha n+\alpha)} f(x - y)dy) =$$

$= \sum_{n=0}^{\infty} (-\lambda)^n \int_0^x \delta(y) f(x-y)dy = \sum_{n=0}^{\infty} (-\lambda)^n \cdot f(x-y)\,|_{y=0} = \frac{1}{1+\lambda} \cdot f(x)$ 21

$y(x) = I_1 + I_2 = \frac{1}{1+\lambda}(y_0 + f(x))$

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Yuqorida $\alpha \to +0$ da yechimning klassik yechim ekanligini ko‘rsatdik. Keyingi navbatda esa $\alpha \to 1+$ da (16) yechimning klassik yechim bo‘lishini ko‘rsatib o‘tamiz:

$\alpha \to 1+$ da Koshi masalasi:

$\begin{cases} y'(x) + \lambda y(x) = f(x) \\ y(0) = y_0 \end{cases}$ 23

Tenglamani yechish uchun Laplas almashtirishlarini bajarib olamiz:

$y(x)\!:\ L\{y(x)\}(s) = \int_0^{\infty} e^{-sx} y(x)dx$

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$y'(x)\!:\ L\{y'(x)\}(s) = \int_0^{\infty} e^{-sx} y'(x) = e^{-sx} y(x)|_0^{\infty} + s\int_0^{\infty} e^{-sx} y(x)dx = 0 - y(0) + sL\{y(x)\}(s) = -y_0 + sL\{y(x)\}(s) + \lambda L\{y(x)\}(s) = L\{f(x)\}(s)$ 25

$L\{y(x)\}(s) = \frac{1}{s+\lambda}(L\{f(x)\}(s) + y_0)$

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$\frac{1}{s+\lambda} \risingdotseq E_{1,1}(-\lambda x) = \sum_{n=0}^{\infty} \frac{(-\lambda x)^n}{n!} = e^{-\lambda x}$

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Bu yerda $\frac{s^{\alpha-\beta}}{s^{\alpha}\pm\omega} \risingdotseq t^{\beta-1} E_{\alpha,\beta}(\mp\omega t^{\alpha})$ teskari Laplas almashtirishi.

$$\frac{1}{s+\lambda} L\{f(x)\}(s) = L\{e^{-\lambda x}\}(s) L\{f(x)\}(s) = L\{f \cdot e^{-\lambda x}\}(s) = L\{\int_0^x f(y) e^{-\lambda(x-y)}\,dy\}(s)$$

$$L\{y(x)\}(s) = y_0 L\{e^{-\lambda x}\}(s) + L\{\int_0^x f(y) e^{-\lambda(x-y)}\,dy\}(s)$$

$y(x) = y_0 e^{-\lambda x} + \int_0^x f(y)\, e^{-\lambda(x-y)} dy$

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(28) yechim 1-tartibli hosila uchun Koshi maslasining klassik yechimi bo‘lishi kelib chiqdi.

Xulosa

Yuqoridagilardan xulosa qilib aytsak (16) yechim bir jinsli bo‘lmagan kasr tartibli tenglama uchun Koshi masalasining Klassik yechimi bo‘lar ekan. Bundan kelib chiqadiki (16) yechimdan biz shu turdagi boshqa tenglamalarni yechishda ham foydalanishimiz mumkin.

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