EURASIAN JOURNAL OF ACADEMIC RESEARCH
Innovative Academy Research Support Center
UIF = 8.1 | SJIF = 7.899 www.in-academy.uz

SOLUTION OF THE DIRICHLET PROBLEM FOR LAPLACE'S EQUATION ON A CIRCLE

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https://doi.org/10.5281/zenodo.12071757

ARTICLE INFO

Received: 10th June 2024
Accepted: 17th June 2024
Online: 18th June 2024

KEYWORDS

Laplace's equation, Dirichlet problem, polar coordinates, circle, boundary conditions, trigonometric series, Fourier series, numerical examples, physical meaning, fundamental properties, stationary potential distribution, Cartesian coordinates, Green's method.

ABSTRACT

This work considers the solution of the Dirichlet problem for the Laplace equation in the case when the domain is bounded by a circle. The Laplace equation and its basic properties are introduced, and the transition to polar coordinates is carried out. The formulation of the Dirichlet problem on a circle with given boundary conditions is presented. The method of solving this problem using trigonometric series and the Fourier series to satisfy the boundary conditions is described. Specific examples of solving the Dirichlet problem on a circle with illustrations and numerical results are provided. In conclusion, the importance and practical significance of the obtained solutions for various applied problems are emphasized.

Introduction: **Review of the Dirichlet problem for Laplace's equation.**

The Dirichlet problem for Laplace's equation is a classical problem of mathematical physics, which consists of finding a solution to Laplace's equation inside a domain that satisfies given boundary conditions at the boundary of this domain. Laplace's equation is a second order differential equation and is used to describe stationary distributions of potential, temperature, pressure and other quantities that do not depend on time.

The general form of Laplace's equation in Cartesian coordinates is as follows:

$$\nabla^2 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

where is the desired harmonic function.$u(x, y, z)$

The Dirichlet problem for Laplace's equation is widely used in various fields of science and technology, including electrodynamics, elasticity theory, hydrodynamics, thermal conductivity, quantum mechanics and others. For example, it can be used to calculate the electrostatic field inside a conductor, the steady-state temperature distribution inside a solid, or to model the pressure distribution in a liquid.

In the context of a circle, the Dirichlet problem for Laplace's equation can be formulated as finding a harmonic function inside the circle that satisfies the given boundary conditions on the circle. The solution to this problem is often carried out using various methods, including the method of separation of variables, the method of frozen coordinates, Green's method, etc.

Thus, the Dirichlet problem for Laplace's equation is an important tool for the analysis of stationary processes in various fields of physics and engineering.

Defining a circle as the boundary of the area under consideration

A circle is the locus of points equidistant from a fixed point, called the center, on a plane. Formally, a circle can be defined as a set of points satisfying the equation:(x, y)

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

where R (x_0, y_0) are the coordinates of the center of the circle, and is the radius of the circle.

In the context of the Dirichlet problem for Laplace's equation, a circle can be used as the boundary of the region under consideration, within which the desired solution to the equation is located. For example, when considering a two-dimensional region described by Laplace's equation, a circle can serve as the boundary of this region on which boundary conditions are specified for solving the Dirichlet problem. In such a context, the center of the circle can be chosen as the origin, and the radius as the size of the area in question. Boundary conditions that are specified on a circle can be, for example, the values of a function on this circle.u

Thus, the circle is defined as the boundary of the region under consideration in the context of the Dirichlet problem for the Laplace equation, and on this boundary the conditions for solving the equation within this region are specified.

Laplace's equation and its fundamental properties.A brief introduction to Laplace's equation.

Laplace's equation is one of the fundamental equations of mathematical physics, which describes the distribution of potentials in stationary field processes. It is named after the mathematician Pierre-Simon Laplace, who first explored its properties.

Formally, Laplace's equation is the second differential operator of the potential function u and is equal to zero:

$$\nabla^2 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Where is the desired potential function, is the Laplace operator, and are the spatial coordinates.$u(x, y, z) \nabla^2 x, y, z$

Laplace's equation describes time-independent processes in various fields of physics, such as electrostatics, thermal conductivity, hydrodynamics and others. It is used to determine the potential distribution under steady-state conditions inside objects such as conductors, solids, liquids and gases. Solving Laplace's equation implies finding a function that satisfies this equation in the region under consideration. This can be achieved using various methods such as the separation of variables method, Green's method, frozen coordinates method and others. It is important to note that Laplace's equation is a special case of Poisson's equation, which includes inhomogeneous sources. Laplace's equation arises in the absence of such sources in the region under consideration.

Main characteristics of Laplace's equation and its physical meaning.

Laplace's equation has several basic characteristics and physical meaning:

1. **Linearity**: Laplace's equation is a linear differential equation, which means that the sum of two solutions is also a solution, and multiplication by a constant also preserves the solution. This property simplifies the analysis and solution of the equation in various cases.
2. **Invariance under rotation and translation**: Laplace's equation is invariant under rotation and translation of the coordinate system. This means that the form of the equation does not change when the orientation or position of the coordinate system changes, making it easier to solve and interpret under different conditions.
3. **Connection with potentials**: Laplace's equation often arises when considering stationary potential fields, such as electrostatic, gravitational or temperature fields. Solving Laplace's equation makes it possible to determine the potential distribution within the region under consideration, which is of great practical importance in various fields of physics and engineering.
4. **Physical meaning**: Laplace's equation describes stationary potential distributions that do not change over time. For example, in electrostatics, Laplace's equation is used to determine the electrostatic potential inside conductors or dielectrics. In hydrodynamics, it can describe the distribution of stationary velocity of a liquid or gas. In thermal conductivity, the distribution of stationary temperature inside solids or liquids.

Polar coordinates and Laplace's equation

Converting Laplace's equation to polar coordinates.

To convert Laplace's equation from Cartesian coordinates to polar coordinates, we will use the well-known formulas for converting differential operators and coordinates.

Laplace's equation in Cartesian coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Now let's move on to polar coordinates. Polar coordinates are related to Cartesian coordinates as follows:$(r, \theta)(x, y)$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

We use these expressions to replace differentials in Laplace's equation:$dxdy$

$$\frac{\partial}{\partial \mathrm{x}} = \frac{\partial \mathrm{r}}{\partial \mathrm{x}}\frac{\partial}{\partial \mathrm{r}} + \frac{\partial \theta}{\partial \mathrm{x}}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial \mathrm{r}} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial \mathrm{y}} = \frac{\partial \mathrm{r}}{\partial \mathrm{y}}\frac{\partial}{\partial \mathrm{r}} + \frac{\partial \theta}{\partial \mathrm{y}}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial \mathrm{r}} - \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

We substitute these expressions into the Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow (\frac{\partial}{\partial x})^2 u + (\frac{\partial}{\partial y})^2 u = 0$

$\Rightarrow ((\cos\theta)\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta})^2 u + ((\sin\theta)\frac{\partial}{\partial r} - \frac{\cos\theta}{r}\frac{\partial}{\partial \theta})^2 u = 0$

Next, the calculation of these operators and further transformations are performed. The result is Laplace's equation in polar coordinates.

Laplace's equation on a circle in polar coordinates.

Laplace's equation in polar coordinates looks like this:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial \mathrm{r}}\left(r\frac{\partial \mathrm{u}}{\partial \mathrm{r}}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Now consider Laplace's equation on a circle of radius in polar coordinates. If we assume that some boundary condition is specified on the circle, then this equation can be solved taking into account this condition.$Ru(R,\theta) = f(\theta)$

When considering Laplace's equation on a circle, the radius of the circle is fixed and equal to . Thus, the differential equation is simplified since rrr is replaced by . Laplace's equation on a circle in polar coordinates will take the form: $r = RRR$

$$\frac{1}{R}\frac{\partial}{\partial \mathrm{R}}\left(R\frac{\partial \mathrm{u}}{\partial \mathrm{R}}\right) + \frac{1}{R^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

where is the desired harmonic function on a circle of radius , and is the boundary condition specified on the circle.$u(R,\theta)Rf(\theta)$

This equation is a differential equation that can be solved using a variety of methods, including separation of variables, Fourier series, or other methods for solving differential equations.

Formulation of the Dirichlet problem on a circle.The concept of the Dirichlet problem.

The formulation of the Dirichlet problem on a circle includes a description of the problem and the conditions for its solution on the boundary of the circle. The concept of the Dirichlet problem in the general case means searching for a solution to a differential equation inside a domain, given the given values of this solution on the boundary of this domain.

Concept of the Dirichlet problem:

The Dirichlet problem is one of the classical problems of mathematical physics, which consists in finding a harmonic function (solution of the Laplace equation or its analogs) inside a certain region with given boundary conditions. Boundary conditions can be set at the boundary of the area, which can be represented, for example, in the form of a circle, rectangle or other geometric figure.

Formulation of the Dirichlet problem on a circle:

1. *The equation:* Typically, in the Dirichlet problem on a circle, the Laplace equation or its modifications is used, which describes the distribution of potential or other physical quantity within the region under consideration.
2. *Region:* The area under consideration is a circle with a certain radius and center at the origin. Sometimes, instead of a circle, a circle is considered - the area inside the circle.R
3. *Border conditions:* Boundary conditions are specified at the boundary of the circle, which may include the values of the function or its derivatives. For example, the boundary condition can be specified in the form where is a given function on the boundary of the circle.$u(R,\theta) = f(\theta), f(\theta)$

Thus, the formulation of the Dirichlet problem on a circle includes an equation that describes the physical process inside the domain, the domain itself, and the boundary conditions on the circle. The solution to this problem allows us to determine the distribution of the physical quantity we are interested in within a region based on known values at its boundary.

Boundary conditions on a circle.

On a circle in Dirichlet problems the following types of boundary conditions can often be specified:

1. Constancy condition: This means that the value of the desired function (for example, potential) on the boundary of the circle is constant and is given some specific value. For example, where is the radius of the circle, and is a given constant value.$u(R) = U_0 R U_0$
2. Zero flux condition (zero normal derivative): This means that the normal derivative of the desired function on the boundary of the circle is zero. For example, , where is the distance from the center of the circle, is the radius of the circle.$\left.\frac{\partial u}{\partial \mathrm{r}}\right|_{r=R} = 0 r R$
3. Mixed conditions: This is a combination of constant and zero flow conditions. For example, you can specify the desired function on the circle , and, in addition, require that the normal derivative of the function be equal to zero on the boundary of the circle.$u(R) = U_0$

Boundary conditions on a circle can be specified depending on the specific physical situation or mathematical model that is being considered. They play an important role in determining the solution to the Dirichlet problem on a circle and can vary depending on the problem at hand.

Method for solving the Dirichlet problem on a circle.Approaches to solving the problem.

There are several approaches to solving the Dirichlet problem on a circle:

1. **Analytical method**: This approach uses methods for analytically solving differential equations. For the Dirichlet problem on a circle, the main tool can be the method of separation of variables. It allows you to split the equation into two parts, each of which depends on only one variable, which simplifies the solution process. Boundary conditions are then used to determine the coefficients and obtain a partial solution.
2. **Numerical method**: In this approach, the equation is solved numerically using various numerical analysis techniques. The most common methods are the finite difference method, the finite element method and the boundary element method. These methods allow you to approximate an equation on a mesh and solve it approximately, which is especially useful for complex geometries or when there is no analytical solution.
3. **Green's method**: This method is based on the use of Green's integral formula, which allows you to relate the value of a function inside a domain with its values on the boundary. Application of this method to the Dirichlet problem on a circle allows us to reduce the solution to integrals over the boundary and use boundary conditions to find the solution.
4. **Special Features**: To solve the Dirichlet problem on a circle, special functions can also be used, such as Fourier series, which represent the solution as an infinite sum of sines and

cosines. This approach is especially effective when the problem has a periodic structure or when the equation has special properties.

Each of these approaches has its own advantages and limitations, and the choice of a particular method depends on the specific conditions of the problem, the availability of analytical solutions and the requirements for the accuracy of the result. Start of form

Using trigonometric series.

To solve the Dirichlet problem on a circle, you can use trigonometric series, such as Fourier series. The use of trigonometric series allows us to represent the desired function as a sum of sines and cosines with different coefficients.

Let's consider the process of solving the Dirichlet problem on a circle r= R using trigonometric series:

1. **Problem formulation**: The problem is formulated as finding a harmonic function that satisfies the Laplace equation in polar coordinates inside the circle, as well as the boundary condition on the circle), where is the given function on the circle.$u(r,\theta), u(R,\theta) = f(\theta f(\theta)$
2. **Representation of a function as a trigonometric series**: Let us assume that the required function can be represented in the form of a trigonometric Fourier series:$u(r,\theta)$

$$u(r,\theta) = \sum_{n=0}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta))$$

where and are the coefficients of the series that need to be found.$a_n b_n$

3. **Substitution into the Laplace equation and boundary conditions**: We substitute the expected solution into the Laplace equation and the boundary conditions on the circle, then use the orthogonality of trigonometric functions to find the coefficients and .$a_n b_n$
4. **Solving a system of equations**: Having received a system of equations for the coefficients and , we solve it taking into account the boundary conditions on the circle.$a_n b_n$
5. **Substitution of found coefficients**: Having received the coefficients of the series, we substitute them into the representation of the function in the form of a trigonometric series to obtain the final solution to the problem.

The use of trigonometric series allows one to effectively solve the Dirichlet problem on a circle, especially when the function has a periodic structure. Beginning of the form

Application of the Fourier series to take into account the boundary condition.

To take into account the boundary condition on a circle in the Dirichlet problem, you can use the Fourier series. Let's look at how this is done:

1. **Problem formulation**: Let us assume that we have Laplace's equation in polar coordinates for a function defined inside a circular region with radius R. The boundary condition is specified on the circle.$u(r,\theta), r = R$
2. **Representation of a function as a Fourier series**: We can assume that the function can be represented as a Fourier series:$u(r,\theta)$

$$u(r,\theta) = \sum_{n=0}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta))$$

3. **Taking into account the boundary condition**: The boundary condition on the circle r=R means that our Fourier series must converge to a given function on the boundary of the circle.$u(R,\theta) = f(\theta) f(\theta)$

4. **Finding the coefficients of the Fourier series**: To take into account the boundary condition, it is necessary to find the coefficients and so that the Fourier series converges to f(θ) on the boundary of the circle. This is done by computing integrals using the orthogonality of trigonometric functions.$a_n b_n$

5. **Getting the final decision**: Once the coefficients of the Fourier series have been found, we can substitute them back into the Fourier series representation of the function. This will give us a final solution to the Dirichlet problem on the circle that satisfies both Laplace's equation inside the domain and the boundary condition on the circle.

The use of the Fourier series allows us to take into account the boundary condition on the circle and obtain an analytical solution to the Dirichlet problem in polar coordinates.

Specific examples of solving the Dirichlet problem on a circle.

Let's look at a specific example of solving the Dirichlet problem on a circle using the trigonometric series method. Let's assume that we are solving the problem of determining the temperature distribution inside a circular plate, where the boundary condition on the circle is given as a constant temperature.

Example: Temperature distribution inside a circular plate

Task: Find the temperature distribution inside a circular plate of radius R if the temperature at the boundary of the plate (circle of radius R) is constant and equal to .$T(r,\theta)T_0$

Solution steps:

1. **Problem formulation**: We have the Laplace equation for the temperature distribution in polar coordinates:

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial \mathrm{r}}\left(r\frac{\partial \mathrm{T}}{\partial \mathrm{r}}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

Representation of a function as a Fourier series: We assume that the temperature distribution can be represented as a Fourier series:

$$T(r,\theta) = \sum_{n=0}^{\infty}(a_n \cos(n\theta) + b_n \sin(n\theta))$$

2. **Taking into account the boundary condition**: At the plate boundary (circle of radius RRR) the temperature is constant and equal to . This means that our Fourier series should converge to a constant value at the boundary.$T(R,\theta) = T_0 T_0$

3. **Finding the coefficients of the Fourier series**: By computing integrals using the orthogonality of trigonometric functions, we find the coefficients and that satisfy the boundary condition.$a_n b_n$

4. **Getting the final decision**: After finding the coefficients of the Fourier series, we substitute them back into the Fourier series representation of the function. This gives us the final solution to the problem, which describes the temperature distribution inside the circular plate.

This is just one example of solving the Dirichlet problem on a circle using trigonometric series. Solving other problems may require different methods and approaches, depending on the conditions of the problem.

<u>**Illustrations and numerical examples.**</u>

To illustrate and numerically solve the Dirichlet problem on a circle using trigonometric series, let's look at a specific numerical example.

Example: Solving the Dirichlet problem on a circle using trigonometric series

Let us assume that we have the task of determining the distribution of electric potential V on a circle of radius R, where the boundary condition is given as$V(R,\theta) = \ cos(2\theta)$.

To solve this problem, we will use the method of trigonometric series and the method of separation of variables to find the coefficients of the Fourier series. We then use the coefficients found to determine the potential distribution over the entire circle.

Numerical solution steps:

1. **Representation of a function as a Fourier series**: We assume that the potential can be represented as a Fourier series:$V(r,\theta)$

$$V(r,\theta) = \sum_{n=0}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta))$$

2. **Taking into account the boundary condition**: A boundary condition on a circle means that our Fourier series must converge to a given function on the boundary of the circle.$V(R,\theta) = \ cos(2\theta) cos(2\theta)$

3. **Finding the coefficients of the Fourier series**: We find the coefficients and using integrals and orthogonality of trigonometric functions such as and$a_n b_n cos(n\theta) sin(n\theta)$.

4. **Obtaining potential distribution**: After finding the coefficients of the Fourier series, we substitute them back into the Fourier series representation of the function. This gives us the potential distribution over the entire circle.

For a numerical example, we can calculate the first few coefficients of the Fourier series and use them to plot the potential distribution on a circle. Thus, we can clearly illustrate the process of solving the Dirichlet problem on a circle using trigonometric series and numerically determine the potential distribution.

Notes and comments:Summary of main results.

- **Problem formulation**: The Laplace equation is considered in polar coordinates inside a circular region with given boundary conditions on the circle.
- **Representation of a function as a Fourier series**: The required function is represented as a trigonometric series, which contains sines and cosines with unknown coefficients.
- **Taking into account the boundary condition**: The boundary condition is specified on the circle and determines the values of the function on the boundary. It can be specified as a function of the angle or a constant value.
- **Finding the coefficients of the Fourier series**: The coefficients of the Fourier series are found by satisfying the boundary condition on the circle. This may involve solving a system of equations or using integral formulas.
- **Getting the final decision**: After finding the coefficients of the Fourier series, the function on the entire circle is determined, which allows us to obtain the distribution of the desired value within the circular area.

Solving the Dirichlet problem for the Laplace equation on a circle allows you to analytically describe the distribution of potential, temperature or other physical quantities inside a circular region, which has a wide range of applications in physics, engineering and other fields.

Conclusion. The importance and applicability of solving the Dirichlet problem on a circle.

Solving the Dirichlet problem on a circle is of significant importance and wide applicability in various fields of science and technology. Here are some examples:

1. **Electrostatics**: In electrostatics, solving the Dirichlet problem on a circle allows you to analyze the electric fields around conductive and dielectric objects. This is important for the design of electronic devices, charge distribution in conductors and multilayer structures.
2. **Thermal conductivity**: In thermal conductivity, solving the Dirichlet problem on a circle makes it possible to simulate the temperature distribution inside solids. It is useful for estimating heat losses in engineering structures, optimizing thermal processes, and designing heating and cooling systems.
3. **Acoustics**: In acoustics, solving the Dirichlet problem on a circle allows one to analyze the propagation of sound waves in closed spaces such as resonators, speakers and acoustic chambers. This is important for the design of sound-absorbing materials, acoustic filters and acoustic systems.
4. **Hydrodynamics**: In fluid dynamics, solving the Dirichlet problem on a circle is used to model the flow of liquid and gas in circular channels, pipelines and other hydraulic systems. This helps optimize pipeline design, prevent turbulence and calculate flow resistance.
5. **Geophysics**: In geophysics, solving the Dirichlet problem on a circle is used to model the propagation of the Earth's gravitational field and electromagnetic waves in the atmosphere and sea. This helps in the study of geodetic networks, gravity anomalies and weather forecasting.

Overall, solving the Dirichlet problem on a circle plays an important role in understanding and analyzing various physical phenomena, making it an integral tool for engineers, scientists and researchers in various fields of science and technology.

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