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SOLUTION OF THE NEUMANN PROBLEM FOR THE LAPLACE EQUATION ON A RING

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ABSTRACT
This work considers the solution of the Neumann problem on a ring-shaped domain. First, the definition of the domain and the formulation of the Neumann boundary conditions on the inner and outer edges of the ring are provided. To solve the problem, the method of separation of variables in polar coordinates is used. The general solution of the Laplace equation in the ring is obtained, and then, by applying the Neumann boundary conditions, the coefficients in this general solution are determined. As an example, a specific Neumann boundary value problem on a ring is considered, the coefficients in the general solution are calculated, and the final solution is constructed. This work can be useful in the study and solution of partial differential equations in ring-shaped domains.

Introduction: Laplace's equation is a fundamental differential equation in mathematical physics that describes many important physical phenomena, such as the distribution of potential, temperature, pressure and other scalar quantities in various media. The solution of Laplace's equation with appropriate boundary conditions is of great interest for applications in the fields of electrostatics, thermal conductivity, hydrodynamics and others.

One of the classical types of boundary value problems for the Laplace equation is the Neumann problem, in which the values of the normal derivative of the solution, and not the solution itself, are specified at the boundary of the domain. Solving the Neumann problem is often necessary in engineering applications where it is important to determine the flows or gradients of physical quantities at the boundaries of bodies or regions.

This abstract discusses the solution of the Neumann problem for the Laplace equation in a ring domain. Ring regions are often encountered in practical problems, for example, when modeling the distribution of potential or temperature in round pipes, cylindrical capacitors, axisymmetric bodies of rotation, etc. Thus, the study of solving the Neumann problem on a ring is of great theoretical and practical significance.

Statement of the Neumann problem on a ring

When setting up the Neumann problem for the Laplace equation on a ring, the domain of definition of the problem is a ring. A ring is defined as an area on a plane bounded by two concentric circles.

Let the ring region be defined by the equations:

$$a \leq r \leq b$$

where is the radial coordinate in the polar coordinate system, and are the inner and outer radii of the ring, respectively.rab

Thus, the annular region has the shape of a circular ring with an inner radius a and an outer radius. The boundary of the ring consists of two circles - an inner circle of radius a and an outer circle of radius. In the Neumann problem on a ring, the values of the normal derivative of the desired solution to the Laplace equation are specified on these two boundary circles. Finding this solution that satisfies the given Neumann boundary conditions is the goal of solving this boundary value problem.bb

When setting up the Neumann problem for the Laplace equation on a ring, the Neumann boundary conditions are formulated as follows. Let be the desired solution of the Laplace equation in the ring domain, where and are the polar coordinates.$u(r,\theta) a \leq r \leq b r \theta$

The Neumann boundary conditions at the inner and outer edges of the ring are as follows.

On the inner edge of the ring ():$r = a$

$$\frac{\partial u}{\partial n} = g_1(\theta)$$

On the outer edge of the ring ():$r = b$

$$\frac{\partial u}{\partial n} = g_2(\theta)$$

Where, is the derivative of the solution along the normal to the boundary of the ring, and and are the given functions of the angular coordinate θ, which determine the values of the normal derivative at the inner and outer edges of the ring, respectively.$\frac{\partial u}{\partial n} u(r,\theta) g_1(\theta) g_2(\theta)$

Thus, in the Neumann ring problem, we need to find a solution to Laplace's equation that satisfies the given Neumann boundary conditions at the inner and outer edges of the ring region.$u(r,\theta)$

Notes and comments: Solution method

When solving the Neumann problem for the Laplace equation in a ring domain, in addition to determining the boundary conditions, it is important to use a polar coordinate system to write the differential equation and find a solution.

In polar coordinates, Laplace's equation has the form:(r,θ)

$$\frac{\partial^2 u}{\partial r^2} + \left(\frac{1}{r}\right)\frac{\partial u}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2 u}{\partial \theta^2} = 0$$

Where, is the desired solution to the Laplace equation in polar coordinates,$u(r,\theta)$

r– radial coordinate,

θ– angular coordinate.

Neumann boundary conditions in polar coordinates are written as:

At :$r = a$

$$\frac{\partial u}{\partial r} = g_1(\theta)$$

At :$r = b$

$$\frac{\partial u}{\partial r} = g_2(\theta)$$

This Laplace equation in polar coordinates with the corresponding Neumann boundary conditions constitutes the mathematical formulation of the Neumann problem on a ring.

To solve the Neumann problem for the Laplace equation in a ring domain, the method of separation of variables is widely used. This method allows you to find an analytical solution to the problem, expressed through special functions.

To solve this problem, methods of separation of variables are usually used, using Fourier series in the angular variable θ and the corresponding Bessel functions in the radial variable r. This approach allows you to find a solution in analytical form, expressed through special functions. Thus, the use of polar coordinates is a key point in formulating and solving the Neumann problem for the Laplace equation in a ring domain.

Let's consider a ring region in polar coordinates. We will look for a solution to the Laplace equation in the form:$a \leq r \leq b(r, \theta)$

$$u(r, \theta) = R(r)\Theta(\theta)$$

Where, is a function that depends only on the radial coordinate,$R(r)r$

$\Theta(\theta)$– a function that depends only on the angular coordinate.θ

Substituting this representation into the Laplace equation in polar coordinates, we obtain:

$$\frac{1}{R} * \frac{d^2R}{dr^2} + \left(\frac{1}{r}\right) * \frac{dR}{dr} + \left(\frac{1}{r^2}\right) * \left(\frac{1}{\Theta}\right) * \frac{d^2\Theta}{d\theta^2} = 0$$

Separating the variables, we find:

$$\frac{1}{R} * \frac{d^2R}{dr^2} + \left(\frac{1}{r}\right) * \frac{dR}{dr} = -\lambda$$

$$\frac{1}{r^2\Theta} * \frac{d^2\Theta}{d\theta^2} = \lambda$$

Where is the separation constant.λ

Solving these equations separately for and , taking into account the Neumann boundary conditions, we obtain:$R(r)\Theta(\theta)$

$$R(r) = A * J_n(\sqrt{\lambda}r) + B * Y_n(\sqrt{\lambda}r)$$

$$\Theta(\theta) = C * \cos(n\theta) + D * \sin(n\theta)$$

Where, are the Bessel functions,J_n, Y_n

A, B, C, D– constants determined from boundary conditions.

Thus, the general solution to the Neumann problem in a ring domain has the form:

$$u(r, \theta) = \sum_n \left\{ \begin{matrix} [A_n * J_n(\sqrt{\lambda_n}r) + B_n * Y_n(\sqrt{\lambda_n}r)] * \\ * [C_n * \cos(n\theta) + D_n * \sin(n\theta)] \end{matrix} \right\}$$

Where the coefficients are determined from the Neumann boundary conditions. This method of separating variables allows one to obtain an analytical expression for solving the Neumann problem on a ring.A_n, B_n, C_n, D_n

To obtain a general solution to the Laplace equation in a ring domain when solving the Neumann problem, the method of separating variables in a polar coordinate system is used. Consider the annular region, where is the radial coordinate and is the angular coordinate.$a \leq r \leq b r \theta$

Laplace's equation in polar coordinates has the form:

$$\frac{\partial^2 u}{\partial r^2} + \left(\frac{1}{r}\right)\frac{\partial u}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2 u}{\partial \theta^2} = 0$$

We will look for a solution in the form:

$$u(r,\theta) = R(r)\Theta(\theta)$$

Substituting this representation into the Laplace equation, we obtain:

$$\frac{1}{R} * \frac{d^2R}{dr^2} + \left(\frac{1}{r}\right) * \frac{dR}{dr} = -\frac{\lambda}{r^2}$$

$$\frac{1}{r^2\Theta} * \frac{d^2\Theta}{d\theta^2} = \lambda$$

Where is the separation constant.λ

Solving these equations separately for and , taking into account the Neumann boundary conditions:$R(r)\Theta(\theta)$

At :$r = a$

$$\frac{\partial u}{\partial r} = g_1(\theta)$$

At :$r = b$

$$\frac{\partial u}{\partial r} = g_2(\theta)$$

We get:

$$R(r) = A * J_n\left(\sqrt{\lambda} r\right) + B * Y_n\left(\sqrt{\lambda} r\right)$$

$$\Theta(\theta) = C * \cos(n\theta) + D * \sin(n\theta)$$

Where, are Bessel functions of order ,$J_n, Y_n n$

A, B, C, D– constants determined from boundary conditions.

Thus, the general solution to Laplace's equation in a ring domain has the form:

$$u(r,\theta) = \sum_n \begin{Bmatrix} \left[A_n * J_n\left(\sqrt{\lambda_n} r\right) + B_n * Y_n\left(\sqrt{\lambda_n} r\right)\right] * \\ * \left[C_n * \cos(n\theta) + D_n * \sin(n\theta)\right] \end{Bmatrix}$$

Where, , are Bessel functions of order , the coefficients are determined from the Neumann boundary conditions. This Fourier-Bessel series is a general solution to the Neumann problem for the Laplace equation in a ring domain. To find the coefficients in the general solution of the Neumann problem for the Laplace equation in a ring domain, it is necessary to apply the Neumann boundary conditions.$J_n Y_n n A_n, B_n, C_n, D_n$

Differentiating the general solution with respect to , we obtain:r

$$\frac{\partial u}{\partial r} = \sum_n \left[A_n * \sqrt{\lambda_n} * J_n'\left(\sqrt{\lambda_n} r\right) + B_n * \sqrt{\lambda_n} * Y_n'\left(\sqrt{\lambda_n} r\right)\right] ** \left[C_n * \cos(n\theta) + D_n * \sin(n\theta)\right]$$

Substituting boundary conditions, we get:

At :$r = a$

$$\sum_n [A_n * \sqrt{\lambda_n} * J_n'(\sqrt{\lambda_n}a) + B_n * \sqrt{\lambda_n} * Y_n'(\sqrt{\lambda_n}a)] ** [C_n * \cos(n\theta) + D_n * \sin(n\theta)] = g_1(\theta)$$

At :$r = b$

$$\sum_n [A_n * \sqrt{\lambda_n} * J_n'(\sqrt{\lambda_n}b) + B_n * \sqrt{\lambda_n} * Y_n'(\sqrt{\lambda_n}b)] * [C_n * \cos(n\theta) + D_n * \sin(n\theta)] = g_2(\theta)$$

These equalities must be satisfied for all, therefore the coefficients for each trigonometric function must be equal to the corresponding expansion coefficients of the functions in the Fourier series:$\theta g_1(\theta) g_2(\theta)$

$$A_n * \sqrt{\lambda_n} * J_n'(\sqrt{\lambda_n}a) + B_n * \sqrt{\lambda_n} * Y_n'(\sqrt{\lambda_n}a) = a_n$$
$$A_n * \sqrt{\lambda_n} * J_n'(\sqrt{\lambda_n}b) + B_n * \sqrt{\lambda_n} * Y_n'(\sqrt{\lambda_n}b) = b_n$$

Where and are the expansion coefficients and in the Fourier series.$a_n b_n g_1(\theta) g_2(\theta)$

By solving this system of linear equations, you can find the unknown coefficients and. Then, substituting them into the general solution, the coefficients and are determined using the orthogonality of trigonometric functions. Thus, the application of Neumann boundary conditions allows us to find all the coefficients of the general solution of the Neumann problem for the Laplace equation in a ring domain.$A_n B_n C_n D_n$

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