
# FURYE QATORI. FUNKSIYALARNI FURYE QATORIGA YOYISH

Usmonov Maxsud Tulqin o'g'li
Toshkent axborot texnologiyalari universiteti
Qarshi filiali 4-kurs talabasi
+99891 947 13 40
maqsudu32@gmail.com
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**MAQOLA TARIXI**

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**KALIT SO'ZLAR**

*Furye qatori, Furye koeffitsiyentlari. Funksiyalarni Furye qatoriga yoyish.*

**ANNOTATSIYA**

*Ushbu maqolada matematikaning eng muhim mavzularidan biri bo'lgan Furye qatori. Funksiyani Furye qatoriga yoyish tog'risida malumot keltirildi va mavjud muanmolar xal etildi. Agar f (x) funksiya [a;b] kesmada monoton bo'lsa yoki [a;b] kesmani chekli sondagi qismiy kesmalarga bo'lish mumkin bo'lsa va bu kesmalarning har birida f (x) funksiya monoton (faqat o'ssa yoki faqat kamaysa) yoki o'zgarmas bo'lsa, f (x) funksiyaga [a;b] kesmada bo'laklimonoton funksiya deyiladi. Agar f (x) funksiya [a;b] kesmada chekli sondagi birinchi tur uzilish nuqtalariga ega bo'lsa, f (x) funksiyaga [a;b] kesmada bo'lakli-uzluksiz funksiya deyiladi. Agar f (x) funksiya [a;b] kesmada uzluksiz yoki bo'lakli-uzluksiz bo'lib, bo'lakli-monoton bo'lsa f (x) funksiya [a;b] kesmada Dirixle shartlarini qanoatlantiradi deyiladi.*

## 1. Furye qatori .

Faraz qilaylik, $f(x)$ funksiya $R=(-\infty,+\infty)$ da berilgan bo'lsin. Ma'lumki, shunday $T\in R\setminus\{0\}$ son topilsaki, $\forall x\in R$ da

$$f(x+T)=f(x)$$

tenglik bajarilsa, $f(x)$ davriy funksiya, $T\neq 0$ son esa uning davri deyiladi.

Agar $T\neq 0$ son $f(x)$ funksiyaning davri bo'lsa, u holda

$$kT \quad (k=\pm1,\pm2,\ldots)$$

sonlar ham shu funksiyaning davri bo'ladi.

Agar $f(x)$ va $g(x)$ davriy funksiyalar bo'lib, $T\neq 0$ ularning davri bo'lsa,

$$f(x)\pm g(x),\quad f(x)\cdot g(x),\quad \frac{f(x)}{g(x)}\quad (g(x)\neq 0)$$

funksiyalar ham davriy bo'lib, ularning davri $T$ ga teng bo'ladi.

$y=\sin x$, $y=\cos x$ funksiyalar $T=2\pi$ davrli funksiya bo'lgan holda ushbu

$\varphi(x)=a\cos\alpha x+b\sin\alpha x$ $(a,b,\alpha-$ o'zgarmas, $\alpha\neq 0)$

funksiya ham davriy funksiya bo'lib, uning davri $T=\dfrac{2\pi}{\alpha}$ bo'ladi. Haqiqatan ham,

$$\varphi\left(x+\frac{2\pi}{\alpha}\right)=a\cos\left[\alpha\left(x+\frac{2\pi}{\alpha}\right)\right]+b\sin\left[\alpha\left(x+\frac{2\pi}{\alpha}\right)\right]=$$
$$=a\cos(\alpha x+2\pi)+b\sin(\alpha x+2\pi)=a\cos\alpha x+b\sin\alpha x=\varphi(x)$$

bo'ladi.

Bu $\varphi(x)=a\cos\alpha x+b\sin\alpha x$ sodda davriy funksiya bo'lib, u garmonika deb ataladi.

Aytaylik, $f(x)$ funksiya $[-\pi,\pi]$ da uzluksiz bo'lsin.

Unda $f(x)\cos nx$, $f(x)\sin nx$ $(n=1,2,3,\ldots)$

funksiyalar ham $[-\pi,\pi]$ da uzluksiz bo‘lib, ular $[-\pi,\pi]$ da integrallanuvchi bo‘ladi. Bu integrallarni quyidagicha belgilaymiz:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx,\quad (n=1,2,\ldots)\quad (1)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx.\quad (n=1,2,\ldots)$$

Bu sonlardan foydalanib, ushbu

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\qquad (2)$$

qatorni ( uni trigonometrik qator deyiladi) hosil qilamiz.

(2) qator funksional qator bo‘lib, uning har bir hadi garmonikadan iborat.

**Ta’rif.** *(2) funksional qator $f(x)$ funksiyaning Furye qatori deyiladi. (1) munosabatlar bilan aniqlangan*

$a_0, a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots$

*sonlar Furye koeffitsiyentlari deyiladi.*

**7.2. Funksiyalarni Furye qatoriga yoyish.**

Demak, berilgan $f(x)$ funksiyaning Furye koeffitsiyentlari shu funksiyaga bog‘liq bo‘lib, (2) formulalar yordamida aniqlanadi, qator esa quyidagicha:

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

belgilanadi.

**1-misol.** *Ushbu* $f(x)=e^{\alpha x}\ (-\pi\le x\le\pi,\ \alpha\ne 0)$ *funksiyaning Furye qatori topilsin.*

◄ (1) formulalardan foydalanib, berilgan funksiyaning Furye koeffitsiyentlarini hisoblaymiz:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} e^{\alpha x}dx=\frac{1}{\alpha\pi}\left(e^{\alpha\pi}-e^{-\alpha\pi}\right)=\frac{2}{\alpha\pi}sh\alpha\pi,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} e^{\alpha x}\cos nxdx=\frac{1}{\pi}\frac{\alpha\cos nx+n\sin nx}{\alpha^2+n^2}e^{\alpha x}\Big|_{-\pi}^{\pi}=$$

$$=(-1)^n\frac{1}{\pi}\cdot\frac{2\alpha}{\alpha^2+n^2}sh\alpha\pi\qquad (n=1,2,\ldots),$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} e^{\alpha x}\sin nxdx=\frac{1}{\pi}\frac{\alpha\sin nx-n\cos nx}{\alpha^2+n^2}e^{\alpha x}\Big|_{-\pi}^{\pi}=$$

$$=(-1)^{n-1}\frac{1}{\pi}\cdot\frac{2n}{\alpha^2+n^2}sh\alpha\pi\qquad (n=1,2,\ldots).$$

Demak,

$f(x)=e^{\alpha x}$ funksiyaning Furye qatori

$$f(x)=e^{\alpha x}\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=$$

$$=\frac{2sh\alpha\pi}{\pi}\left[\frac{1}{2\alpha}+\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2+n^2}(\alpha\cos nx-n\sin nx)\right]$$

bo‘ladi.►

Aytaylik, $f(x)$ funksiya $[-\pi,\pi]$ da berilgan juft funksiya bo‘lsin: $f(-x)=f(x)$. U holda $f(x)\cdot\cos nx$ juft, $f(x)\cdot\sin nx$ toq $(n=1,2,3,\ldots)$ funksiya bo‘ladi.

(1) formulalardan foydalanib, $f(x)$ funksiyaning Furye koeffitsiyentlarini topamiz:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nxdx+\int_{0}^{\pi} f(x)\cos nxdx\right]=$$

$$=\frac{1}{\pi}\left[\int_{0}^{\pi} f(x)\cos nxdx+\int_{0}^{\pi} f(x)\cos nxdx\right]=$$

$$\frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nxdx\quad (n=0,1,2,\ldots).$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nxdx+\int_{0}^{\pi} f(x)\sin nxdx\right]=$$

$$=\frac{1}{\pi}\left[-\int_{0}^{\pi} f(x)\sin nxdx+\int_{0}^{\pi} f(x)\sin nxdx\right]=0\qquad (n=1,2,\ldots).$$

Demak, juft $f(x)$ funksiyaning Furye koeffitsiyentlari

$$a_n=\frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nxdx\quad (n=0,1,2,\ldots)$$

$$b_n=0\quad (n=1,2,\ldots)$$

bo‘lib, Furye qatori

$$f(x)\sim \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$$

bo‘ladi.

Aytaylik, $f(x)$ funksiya $[-\pi,\pi]$ da berilgan toq funksiya bo‘lsin:

$f(-x)=-f(x)$. U holda

$f(x)\cdot\cos nx$ toq, $f(x)\cdot\sin nx$ juft $(n=1,2,3,\ldots)$ funksiya bo‘ladi.

(1) formulalardan foydalanib, $f(x)$ funksiyaning Furye koeffitsiyentlarini topamiz:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nxdx+\int_{0}^{\pi} f(x)\cos nxdx\right]=$$

$$=\frac{1}{\pi}[-\int_{0}^{\pi} f(x)\cos nxdx+f(x)\cos nxdx]=0 \qquad (n=0,1,2,\ldots),$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nxdx+\int_{0}^{\pi} f(x)\sin nxdx\right]=$$

$$=\frac{2}{\pi}\left[\int_{0}^{\pi} f(x)\sin nxdx\right] \qquad (n=1,2,\ldots).$$

Demak, toq $f(x)$ funksiyaning Furye koeffitsiyentlari

$$a_n=0, \quad (n=0,1,2,\ldots),$$

$$b_n=\frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nxdx, \ (n=1,2,\ldots)$$

bo‘lib, Furye qatori

$$f(x)\sim\sum_{n=1}^{\infty} b_n \sin nx$$

bo‘ladi.

**2-misol.** *Ushbu*

$$f(x)=x^2 \quad (-\pi\le x\le\pi)$$

*juft funksiyaning Furye qatori topilsin.*

◀ Avvalo berilgan funksiyaning Furye koeffitsiyentlarini topamiz:

$$a_0=\frac{2}{\pi}\int_{0}^{\pi} x^2dx=\frac{2}{3}\pi^2,$$

$$a_n=\frac{2}{\pi}\int_{0}^{\pi} x^2\cos nxdx=\frac{2}{\pi}x^2\frac{\sin nx}{n}\Big|_0^{\pi}-\frac{4}{n\pi}\int_{0}^{\pi} x\sin nxdx=$$

$$=\frac{4}{\pi n}\left(\frac{x\cos nx}{n}\Big|_0^{\pi}-\frac{1}{n}\int_{0}^{\pi}\cos nxdx\right)=(-1)^n\cdot\frac{4}{n^2}.\ (n=1,2,\ldots)$$

Demak, $f(x)=x^2$ funksiyaning Furye qatori

$$f(x)=x^2\sim\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$$

bo‘ladi. ▶

**3-misol.** *Ushbu*

$$f(x)=x \quad (-\pi\le x\le\pi)$$

*toq funksiyaning Furye qatori topilsin.*

◀Berilgan funksiyaning Furye koeffitsiyentlarini hisoblaymiz:

$$b_n=\frac{2}{\pi}\int_{0}^{\pi} x\sin nxdx=\frac{2}{\pi}\left(-\frac{x\cos nx}{n}\Big|_0^{\pi}+\frac{1}{n}\int_{0}^{\pi}\cos nxdx\right)=\frac{2(-1)^{n-1}}{n}$$

.

Demak, $f(x)=x$ funksiyaning Furye qatori

$$f(x)\sim\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2}{n}\sin nx$$ bo‘ladi.▶

Faraz qilaylik, $f(x)$ funksiya $[-p,p]$ $(p>0)$ segmentda uzluksiz bo‘lsin. Ma’lumki, ushbu

$$t=\frac{\pi}{p}x$$

almashtirish $[-p,p]$ oraliqni $[-\pi,\pi]$ ga o‘tkazadi, ya’ni $x$ o‘zgaruvchi $[-p,p]$ da o‘zgarganda $t$ o‘zgaruvchi $[-\pi,\pi]$ da o‘zgaradi. Endi

$$f(x)=f\left(\frac{p}{\pi}t\right)=\varphi(t).$$

deymiz. Unda $\varphi(t)$ funksiya $[-\pi,\pi]$ oraliqda berilgan uzluksiz funksiya bo‘ladi. Bu funksiyaning Furye koeffitsiyentlari

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\cos ntdt, \ (n=0,1,2,\ldots)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\sin ntdt \quad (n=1,2,\ldots)$$

ni topib, Furye qatorini yozamiz:

$$\varphi(t)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt).$$

Modomiki,

$$t=\frac{\pi}{p}x$$

ekan, unda

$$\varphi\left(\frac{\pi}{p}x\right)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\frac{\pi}{p}x+b_n\sin n\frac{\pi}{p}x\right),$$

bo‘lib, uning koeffitsiyentlari

$$a_n=\frac{1}{p}\int_{-p}^{p}\varphi\left(\frac{\pi}{p}x\right)\cos n\frac{\pi}{p}xdx,\quad (n=0,1,2\ldots)$$

$$b_n=\frac{1}{p}\int_{-p}^{p}\varphi\left(\frac{\pi}{p}x\right)\sin n\frac{\pi}{p}xdx.\quad (n=1,2\ldots)$$

bo‘ladi. Natijada $[-p,p]$ da berilgan $f(x)$ funksiyaning Furye qatorini quyidagicha

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p}+b_n\sin\frac{n\pi x}{p}\right)$$

bo‘lishini topamiz, bunda

$$a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi x}{p}dx\quad (n=0,1,2\ldots)$$

$$b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}xdx\quad (n=1,2\ldots)$$

**4-misol.** *Ushbu*

$$f(x)=e^x\quad (-1\le x\le 1)$$

*funksiyaning Furye qatori topilsin.*

◄ Yuqoridagi formulalardan foydalanib, $f(x)=e^x$ funksiyaning Furye koeffitsiyentilarini topamiz:

$$a_0=\int_{-1}^{1}e^xdx=e-e^{-1},\ \ a_n=\int_{-1}^{1}e^x\cos n\pi xdx=\frac{n\pi\sin n\pi x-\cos n\pi x}{1+n^2\pi^2}e^x\Big|_{-1}^{1}=$$

$$=\frac{1}{1+n^2\pi^2}\left(e\cos n\pi-e^{-1}\cos n\pi\right)=(-1)^n\frac{e-e^{-1}}{1+n^2\pi^2}\quad (n=1,2,\ldots),$$

$$b_n=\int_{-1}^{1}e^x\cos n\pi xdx=\frac{\sin n\pi x-n\pi\cos n\pi x}{1+n^2\pi^2}e^x\Big|_{-1}^{1}=$$

$$=\frac{1}{1+n^2\pi^2}\left(en\pi\cos n\pi+n\pi e^{-1}\cos n\pi\right)=$$

$$=\frac{n\pi(-1)^n}{1+n^2\pi^2}\left(e^{-1}-e\right)=(-1)^{n+1}\frac{e-e^{-1}}{1+n^2\pi^2}\quad (n=1,2,\ldots)$$

Demak,

$$f(x)=e^x\ \ (-1\le x\le 1)$$

funksiyaning Furye qatori

$$e^x\sim\frac{e-e^{-1}}{2}+\left(e-e^{-1}\right)\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{1+n^2\pi^2}\cos n\pi+\frac{(-1)^{n+1}}{1+n^2\pi^2}n\pi\sin n\pi x\right]$$

bo‘ladi.►

Aytaylik, $f(x)$ funksiya $[a,b]$ da berilgan bo‘lsin. $[a,b]$ segment $a_k$ nuqtalar yordamida bo‘laklarga ajratilgan. $(a_0=a,\ a_n=b)$.

Agar har bir $(a_k,a_{k+1})$ $(k=0,1,2,\ldots,n-1)$ da $f(x)$ funksiya differensiallanuvchi bo‘lib, $x=a_k$ nuqtalarda chekli o‘ng

$$f'(a_k+0)\quad (k=0,1,2,\ldots,n-1),$$

va chap

$$f'(a_k-0)\quad (k=0,1,2,\ldots,n)$$

hosilalarga ega bo‘lsa, $f(x)$ funksiya $[a,b]$ da bo‘lakli-differensiallanuvchi deyiladi.

Endi Furye qatorining yaqinlashuvchi bo‘lishi haqidagi teoremani isbotsiz keltiramiz.

**Teorema.** $2\pi$ *davrli* $f(x)$ *funksiya* $[-\pi,\pi]$ *oraliqda bo‘lakli-differensiallanuvchi bo‘lsa, u holda bu funksiyaning Furye qatori*

$$f(x)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos kx+b_k\sin kx\right)$$

$[-\pi,\pi]$ *da yaqinlashuvchi bo‘lib, uning yig‘indisi*

$$\frac{f(x+0)+f(x-0)}{2}$$

*ga teng bo‘ladi.*

**5-misol.** *Ushbu*

$$f(x)=\cos ax\quad (-\pi\le x\le\pi\ ,\ a\ne n\in Z)$$

*funksiyaning Furye qatori topilsin va u yaqinlashishga tekshirilsin.*

◄ Bu funksiyaning Furye koeffitsiyentlarini topamiz. Qaralayotgan funksiya juft bo‘lgani uchun

$$b_n=0\quad (n=1,2,3,\ldots)$$

bo‘lib,

$$a_n = \frac{2}{\pi}\int_0^{\pi}\cos ax\cos nxdx = \int_0^{\pi}\left[\cos(a-n)x+\cos(a+n)x\right]dx =$$

$$= \frac{\sin a\pi}{\pi}(-1)^n\left[\frac{1}{a+n}+\frac{1}{a-n}\right]$$

bo‘ladi. Demak,

$$f(x)\sim\frac{\sin a\pi}{\pi}\left[\frac{1}{a}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{a+n}+\frac{1}{a-n}\right)\cos nx\right]$$

.

Agar $f(x)=\cos ax$ funksiya teoremaning shartlarini bajarishini e’tiborga olsak, unda

$$\cos ax = \frac{\sin a\pi}{\pi}\left[\frac{1}{a}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{a+n}+\frac{1}{a-n}\right)\cos nx\right]$$

bo‘lishini topamiz.►

## Foydalanilgan adabiyotlar:

1. Данко II.И, Попов А.Г. Кожевников Т.Я. Высшая математика в упражнениях и задачах: В 2 ч. М. Высш. I I I К 1966. Ч 1-2.
2. Романовский II. И Ряды Фурье. Теория поля. Аналитеческие и специалные функции. Преобразования Лапласа. М.: Наука, 1973 г.
3. Гмурман В. Н. Эҳтимоллар назарияси ва математик статистика. Тошкент, «Ўқитувчи», 1978
4. Н.М.Жабборов, Е.О.Аликулов, Қ.С.Ахмедова Олий математика. 1-2-қисм . Қарши 2010
5. Гнеденко В. Курс теории вероятностей и математической статистики. М., Высшая школа, 1981.
6. Sirojiddinov S.X., Mamatov M. Ehtimollar nazariyasi kursi. T. O‘qituvchi, 1980.
7. Беклимишсв Д.В. Курс аналитической геометрии и линейной алгебры. М. Наука, 1964.
8. Берман Г.Н Сборник задач по курсу математического анализа. М . Наука, 1965.
9. Бугров Я.С Никольский С.М Элементы линейнойалгебры и аналитической геометрии. М. Наука, 1988.
10. Бугров Я.С Никольский С.М Дифференциалные уравнения. Кратные интегралы. Ряды. Фурье. М. Наука 1961, 1985.
11. Минорский В.И. Сборник задач по высшей математике. М: Наука, 1987.
12. Пискунов Н.С. Дифференциальное исчисления для втузов. М. Наука, 1985. Т. 1-2.
13. Чистяков В.П. Курс теории вероятностей. М.: Наука, 1982.