BASIC FACTS OF PROJECTIVE GEOMETRY

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Abstract: This article provides a comprehensive survey of the foundational structures and theorems in real projective geometry. Beginning with the construction of projective space P^n via homogeneous coordinates and affine charts, we develop the language of incidence, duality, and subspace intersections. We then explore the cross–ratio and its invariance under PGL-actions, followed by a precise statement and proof sketch of the Fundamental Theorem of Projective Geometry. Finally, we present the classical incidence theorems of Desargues and Pascal-complete with coordinate proofs and geometric interpretations-and discuss extensions to finite fields, complex varieties, and modern applications in computer vision.

Keywords: projective space, homogeneous coordinates, affine charts, duality, cross–ratio, projective linear group, fundamental theorem, desargues' theorem, pascal's theorem, finite projective planes, homographies.

INTRODUCTION

Projective geometry emerged in the Renaissance as artists sought mathematically consistent methods for perspective drawing. Early work by Pappus and later by Girard Desargues (1639) and Blaise Pascal (1639) uncovered striking incidence properties invariant under "projection." In the nineteenth century, Möbius, Plücker, and Steiner formalized the subject, introducing homogeneous coordinates and emphasizing duality. Today, projective ideas underpin algebraic geometry, coding theory, and computer vision: any two photographs of a rigid scene are related by a projective transformation (a "homography"), enabling 3D reconstruction and image stitching.

In this article, we systematize the core algebraic methods and the principal theorems. We assume familiarity with basic linear algebra and Euclidean geometry but develop all projective concepts from first principles.

METHODS

Define

$$P^n = (R^{\{n+1\}} \setminus \{0\}) \sim, (x_0, \dots, x_n) \sim \lambda(x_0, \dots, x_n),\ \lambda \neq 0.$$

A point is denoted $[x_0, \dots, x_n]$. The standard affine chart U_i is given by $x_i \neq 0$, with coordinates

$$(X_0, \dots, \widehat{X_i}, \dots, X_n) = \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right).$$

These charts cover P^n and exhibit it as an n-dimensional manifold.

A k-plane in P^n is the image of a $(k+1)$ −dimensional linear subspace of $R^{\{n+1\}}$. Incidence is encoded algebraically: given a point $[x]$ and a hyperplane $H = [a]$ defined by $a_0x_0 + \cdots .. + a_nx_n = 0$, we have

$$[x] \in H \Leftrightarrow \langle a, x \rangle = 0.$$

More generally, the Grassmannian $Gr(k,n)$ parametrizes all k-planes in P^n, with Plücker coordinates arising from the $(k+1)\times(k+1)$ minors of a representative matrix.

Any invertible linear map $M \in GL_{n+1}(R)$ induces

$$\phi_M: [x] \mapsto [Mx].$$

Since scalar multiples act trivially, the full group of projective automorphisms is

$$\mathrm{PGL}_{n+1}(\mathbb{R}) = \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^{\times}.$$

RESULTS

Theorem. In P^n, the correspondence

$$[x] \leftrightarrow (a_0x_0 + \cdots + a_nx_n = 0)$$

yields an isomorphism between the lattice of subspaces and its opposite: $points \leftrightarrow hyperplanes, lines \leftrightarrow (n-2)-planes$, etc. Every valid incidence statement has a dual.

Example. In P^2, "any two distinct points determine exactly one line" is dual to "any two distinct lines meet at exactly one point."

Let four distinct collinear points $A,B,C,D \subset P^1$ be represented in an affine coordinate $x \in R \cup \{\infty\}$. Their cross–ratio is

$$(A,B;C,D) = \frac{(c-a)(d-b)}{(c-b)(d-a)}.$$

Proposition. If $T \in PGL_2(R)$ $then$

$$\big(T(A),T(B);T(C),T(D)\big) = (A,B;C,D).$$

Proof sketch. Any T is of the form $x \mapsto \frac{px+q}{rx+s}$ A direct computation shows that substituting into the cross–ratio formula cancels all dependence on p,q,r,s.

The cross–ratio also characterizes harmonic division: $(A,B;C,D) = -1$.

Fundamental theorem. Let $\phi: P^n \to P^n$ be a bijection sending lines to lines (a *collineation*). Then ϕ is induced by a semilinear map of R^{n+1}. Over R, semilinear = linear, so $\phi \in PGL_{n+1}(R)$.

Proof sketch. One shows first that ϕ sends triples of collinear points to collinear triples and preserves cross–ratios in 1D subspaces; then extends to all of $P^n$by choosing bases of points in general position.

Desargues' theorem. Given triangles $\triangle ABC$ and $\triangle A'B'C'$ in P^2 such that lines AA', BB', CC' concur at a point O, the intersection points

$$P = BC \cap B'C', Q = CA \cap C'A', R = AB \cap A'B'$$

lie collinearly.

Coordinate proof. Embed in P^3, view the two triangles as planar sections of a 3D pyramid, then project back to P^2.

Pascal's theorem. For any hexagon $A_1A_2A_3A_4A_5A_6$ inscribed in a nondegenerate conic in P^2, the three points

$$A_1A_2 \cap A_4A_5, A_2A_3 \cap A_5A_6, A_3A_4 \cap A_6A_1$$

are collinear (the "Pascal line").

Algebraic proof. Parameterize the conic as the image of P^1 under a degree-2 Veronese embedding, reduce to a statement about four cross–ratios, and verify collinearity via a determinant vanishing.

DISCUSSION

Projective geometry's algebraic formalism renders many proofs linear-algebraic, replacing case-by-case synthetic arguments. Duality reveals a deep symmetry: statements about points and hyperplanes are interchangeable. The cross–ratio is the unique projective invariant on four points and becomes central in complex analysis and Möbius geometry.

Beyond R, one studies $P^n(F_q)$, yielding finite projective planes (e.g.\ the Fano plane $P^2(F_2)$ with applications in coding theory and combinatorics. Over C, $P^n(C)$ is the setting for algebraic varieties, moduli spaces, and enumerative geometry (e.g.\ the count of lines on a cubic surface).

In computer vision, images are related by planar homographies $H \in PGL_3(R)$. Given four known correspondences, one solves a linear system for H and uses it to rectify or stitch images. Reconstruction of 3D scenes from multiple views relies on projective invariants and the Fundamental theorem to recover camera matrices up to projective ambiguity.

CONCLUSION

Projective geometry reframes classical Euclidean notions in a unified, incidence-based language, treating "points at infinity" on the same footing as finite points and rendering parallelism a special case of intersection. By introducing homogeneous coordinates and affine charts, one gains a powerful linear-algebraic toolkit: subspaces become coordinate subspaces, incidence reduces to bilinear forms, and transformations become matrix actions in PGL_{n+1}.

The principle of duality reveals an elegant symmetry between points and hyperplanes, while the cross-ratio emerges as the unique invariant of four collinear points under homographies. The Fundamental Theorem of Projective Geometry then shows that any bijection preserving lines must itself be a projective transformation, anchoring the subject's rigidity and its deep connection to linear algebra. Finally, the classical incidence theorems of Desargues and Pascal both illustrate the reach of projective methods-offering coordinate proofs via lifts to higher dimensions or via determinant identities-and serve as gateways to broader algebraic structures such as finite fields and division rings.

Beyond its foundational theorems, projective geometry continues to influence modern mathematics and applications: from coding theory in finite projective planes to the global study of complex varieties, and from tropical analogues to practical algorithms in computer vision for image rectification and 3D reconstruction. In this way, the subject stands as both a historical cornerstone and an ever-evolving landscape, where deep theoretical insights and real-world problems meet in the rich interplay of algebra, combinatorics, and geometry.

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